Łukasiewicz Logic and Automata over MV-Algebras

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Outline

- weighted logic over arbitrary semirings:
  characterization of recognizable series
- many-valued logics:
  general-purpose extension of classical logic
- Łukasiewicz logic
  special fragment $\mathrm{MSO}^{(\overline{Ł},w)}$ for words
- MV-algebras – truth domains for Łukasiewicz logic
- MV-semirings derived from MV-algebras
- weighted logics over MV-semiring $\sim \mathrm{MSO}^{(\overline{Ł},w)}$
- formal series definable in $\mathrm{MSO}^{(\overline{Ł},w)}$ iff
  recognizable over MV-semiring

Łukasiewicz Logic – Syntax

(monadic second order)

set W of truth values (often $[0,1] \subseteq \mathbb{R}$)

truth constants syntactic representatives for elements of W

connectives $\neg, \&, \underline{\vee}$ (strong), $\vee, \wedge$ (weak)

quantifiers $\forall, \exists$ $(\forall_{Ł}, \exists_{Ł})$

signature Σ of relation symbols with arities

variables $\mathbb{X} = \mathbb{X}_1 \cup \mathbb{X}_2$ (first- and second-order)

atoms $p(x_1, \ldots, x_n)$, n-ary $p \in \Sigma$, $x_i \in \mathbb{X}_1$
$X(x)$, $X \in \mathbb{X}_2$, $x \in \mathbb{X}_1$

formulas $\varphi ::= c \mid P \mid \neg\varphi \mid \varphi * \psi \mid Qx\varphi$
where $c \in W$, P is an atom, φ, ψ are formulas,
$* \in \{\&, \underline{\vee}, \vee, \wedge\}$, $Q \in \{\forall, \exists, \forall_{Ł}, \exists_{Ł}\}$ and $x \in \mathbb{X}$

Many valued logic – semantics of atoms

set W of truth values

W-relation $R : S^n \longrightarrow W$, n-ary on set S

W-Σ-structure $\mathcal{S} = (S, [\cdot]_{\mathcal{S}})$ with

- domain $S \neq \emptyset$
- for every n-ary $p \in \Sigma$
  a W-relation $[p]_{\mathcal{S}} : S^n \longrightarrow W$

$\mathbb{X}$-S-assignment $\sigma : \mathbb{X}_1 \longrightarrow S$,
$\sigma : \mathbb{X}_2 \longrightarrow (S \longrightarrow W)$ (unary W-relation)

W-Σ-interpretation $(\mathcal{S}, \sigma)$ with W-Σ-structure $\mathcal{S}$ and
$\mathbb{X}$-S-assignment σ

truth value of atomic formulas in $(\mathcal{S}, \sigma)$:

$$\begin{aligned} [\![p(x_1, \ldots, x_n)]\!]_{(\mathcal{S},\sigma)} &= [p]_{\mathcal{S}}(\sigma(x_1), \ldots, \sigma(x_n)) \\ [\![X(x)]\!]_{(\mathcal{S},\sigma)} &= \sigma(X)(\sigma(x)) \end{aligned}$$

Łukasiewicz logic $\mathrm{MSO}^{(Ł,W)}(A, \mathbb{X})$ on words

Logic $\mathrm{MSO}^{(Ł,W)}(A, \mathbb{X})$ parameterized by
alphabet A, set W of truth values, set $\mathbb{X}$ of variables

fixed set of relation symbols: binary $\leq$, unary $\{P_a \mid a \in A\}$

every word $w \in A^*$ defines

positions $\mathrm{pos}(w) = \{0, \ldots, |w|\}$ between letters in w

word structure $\underline{w} = (\mathrm{pos}(w), [\cdot]_w)$ where

$$[\leq]_w (i, j) = \begin{cases} 1 & \text{iff } i \leq j \\ 0 & \text{otherwise} \end{cases}$$

$$[P_a]_w (i) = \begin{cases} 1 & \text{iff } i > 0 \text{ and } w_i = a \\ 0 & \text{otherwise} \end{cases}$$

Semantics of atoms in $\mathrm{MSO}^{(Ł,W)}(A,\mathbb{X})$

interpretation $(\underline{w},\sigma)$ with

assignment $\sigma : \mathbb{X}_1 \longrightarrow \mathrm{pos}(w)$ and
$\sigma : \mathbb{X}_2 \longrightarrow (\mathrm{pos}(w) \longrightarrow \{0,1\})$

truth value of atomic formulas in $(\underline{w},\sigma)$:

$$
\begin{aligned}
[\![x \le y]\!]_{(\underline{w},\sigma)} &= \begin{cases} 1 & \text{iff } \sigma(x) \le \sigma(y) \\ 0 & \text{otherwise} \end{cases} \\
[\![P_a(x)]\!]_{(\underline{w},\sigma)} &= \begin{cases} 1 & \text{iff } \sigma(x) > 0 \text{ and } w_{\sigma(x)} = a \\ 0 & \text{otherwise} \end{cases} \\
[\![X(x)]\!]_{(\underline{w},\sigma)} &= \sigma(X)(\sigma(x)) \in \{0,1\}
\end{aligned}
$$

all atoms are crisp (take only truth values in $\{0,1\}$)

MV-algebras

$(W, \oplus, \otimes, \neg, 0_W, 1_W)$ where

- $(W, \oplus, 0_W)$ is a commutative monoid,
- for all $x \in W : x \oplus 1_w = 1_w$,
- $\neg 0_W = 1_W$ and $\neg 1_W = 0_W$,
- for all $x, y \in W : \neg(\neg x \oplus \neg y) = x \otimes y$,
- for all $x, y \in W : x \oplus (\neg x \otimes y) = y \oplus (\neg y \otimes x)$

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ordering on W $\quad x \leq y \quad$ iff $\quad \neg x \oplus y = 1$

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lattice operations on $(W, \leq, 0, 1)$ satisfy

$$\begin{aligned} x \vee y &= x \oplus (\neg x \otimes y) \\ x \wedge y &= x \otimes (\neg x \oplus y) \end{aligned}$$

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distributivity:

$$\begin{aligned} x \otimes (y \vee z) &= (x \otimes y) \vee (x \otimes z) \\ x \oplus (y \wedge z) &= (x \oplus y) \wedge (x \oplus z) \end{aligned}$$

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$\forall x : \neg\neg x = x$,

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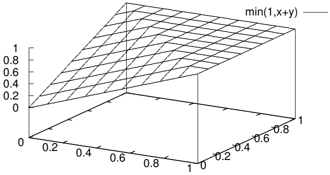
$\forall x : \neg\neg x = x$, in general $\oplus, \otimes$ not idempotent

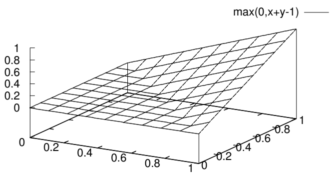
Examples

for $a, b \in \mathbb{R}$ where $a < b$, the structure $([a, b], \oplus, \otimes, \neg, a, b)$ where for all $x, y \in [a, b]$

$$\neg x = a + b - x$$

$$x \oplus y = \min\{b, x + y - a\} \qquad x \otimes y = \max\{a, x + y - b\}$$





standard MV-algebra $[0, 1]^{Ł} = ([0, 1], \oplus, \otimes, \neg, 0, 1)$

isomorphism $f : [0, 1] \longrightarrow [a, b]$ where $f(x) = a + x(b - a)$

More examples

$[0,1]^{Ł} = ([0,1], \oplus, \otimes, \neg, 0, 1)$ where

$$\neg x = 1 - x$$

$$x \oplus y = \min\{1, x + y\} \qquad x \otimes y = \max\{0, x + y - 1\}$$

- sub-MV-algebra $([0,1] \cap \mathbb{Q}, \oplus, \otimes, \neg, 0, 1)$
- finite MV-algebra $\left(\left\{\frac{i}{n} \mid i \in \{0, \ldots, n\}\right\}, \oplus, \otimes, \neg, 0, 1\right)$ for $n \in \mathbb{N} \setminus \{0\}$
- finite MV-algebra $(\{0, \ldots, n\}, \oplus, \otimes, \neg, 0, n)$ for $n \in \mathbb{N} \setminus \{0\}$
- boolean algebra $(\{0, 1\}, \max, \min, \neg, 0, 1)$

More examples

$[0,1]^{Ł} = ([0,1], \oplus, \otimes, \neg, 0, 1)$ where

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Theorem (Chang 1958)

An equation holds in every MV-algebra iff it holds in $[0,1]^{Ł}$.

Łukasiewicz logic – semantics of formulas

truth domain = MV-algebra $\mathbb{W} = (W, \oplus, \otimes, \neg, 0_W, 1_W)$

truth value of non-atomic formulas in $(\mathcal{S}, \sigma)$:

$$[\![\neg\varphi]\!]_{(\mathcal{S},\sigma)} = \neg\, [\![\varphi]\!]_{(\mathcal{S},\sigma)}$$

$$[\![\varphi \veebar \psi]\!]_{(\mathcal{S},\sigma)} = [\![\varphi]\!]_{(\mathcal{S},\sigma)} \oplus [\![\psi]\!]_{(\mathcal{S},\sigma)} \qquad [\![\varphi \,\&\, \psi]\!]_{(\mathcal{S},\sigma)} = [\![\varphi]\!]_{(\mathcal{S},\sigma)} \otimes [\![\psi]\!]_{(\mathcal{S},\sigma)}$$

$$[\![\varphi \vee \psi]\!]_{(\mathcal{S},\sigma)} = [\![\varphi]\!]_{(\mathcal{S},\sigma)} \vee [\![\psi]\!]_{(\mathcal{S},\sigma)} \qquad [\![\varphi \wedge \psi]\!]_{(\mathcal{S},\sigma)} = [\![\varphi]\!]_{(\mathcal{S},\sigma)} \wedge [\![\psi]\!]_{(\mathcal{S},\sigma)}$$

$$[\![\exists x \varphi]\!]_{(\mathcal{S},\sigma)} = \bigvee_{i \in I} [\![\varphi]\!]_{(\mathcal{S},\sigma[x \mapsto i])} \qquad [\![\forall x \varphi]\!]_{(\mathcal{S},\sigma)} = \bigwedge_{i \in I} [\![\varphi]\!]_{(\mathcal{S},\sigma[x \mapsto i])}$$

$$[\![\exists_{Ł} x \varphi]\!]_{(\mathcal{S},\sigma)} = \bigoplus_{i \in I} [\![\varphi]\!]_{(\mathcal{S},\sigma[x \mapsto i])} \qquad [\![\forall_{Ł} x \varphi]\!]_{(\mathcal{S},\sigma)} = \bigotimes_{i \in I} [\![\varphi]\!]_{(\mathcal{S},\sigma[x \mapsto i])}$$

for finite $S \cup W^S$, $I = S$ for $x \in \mathbb{X}_1$ and $I = W^S$ for $x \in \mathbb{X}_2$

Semantics of sentences in $\mathrm{MSO}^{(Ł,W)}(A, \mathbb{X})$

truth domain = MV-algebra $\mathbb{W} = (W, \oplus, \otimes, \neg, 0_W, 1_W)$

semantics of formulas $\varphi \in \mathrm{MSO}^{(Ł,W)}(A, \mathbb{X})$
according to truth functions

$$\vee \mapsto \max \quad \& \mapsto \otimes \quad \exists \mapsto \max \quad \forall_Ł \mapsto \bigotimes$$

every sentence $\varphi \in \mathrm{MSO}^{(Ł,W)}(A, \mathbb{X})$ defines mapping

$$s_\varphi : A^* \longrightarrow W \quad \text{where} \quad s_\varphi(w) = [\![\varphi]\!]_{\underline{w}}$$

MV-semirings

MV-algebra $\mathbb{W} = (W, \oplus, \otimes, \neg, 0_W, 1_W)$ with lattice operations $x \vee y = x \oplus (\neg x \otimes y), x \wedge y = x \otimes (\neg x \oplus y)$

[B. Gerla: Automata over MV-Algebras, ISMVL'04]

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$\mathbb{W}_\vee = (W, \vee, \otimes, 0_W, 1_W)$ is a semiring

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$\mathbb{W}_\wedge = (W, \wedge, \oplus, 1_W, 0_W)$ is a semiring

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$\mathbb{W}_\wedge = (W, \wedge, \oplus, 1_W, 0_W)$ is a semiring

$\neg$ is isomorphism between $\mathbb{W}_\vee$ and $\mathbb{W}_\wedge$

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both semirings are commutative and idempotent

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$([0, 1], \vee, \otimes, \neg, 0, 1)$ is locally finite.

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$\mathbb{W}_\wedge = (W, \wedge, \oplus, 1_W, 0_W)$ is a semiring

$\neg$ is isomorphism between $\mathbb{W}_\vee$ and $\mathbb{W}_\wedge$

both semirings are commutative and idempotent

$([0,1], \vee, \otimes, \neg, 0, 1)$ is locally finite.

for $W = [0,1]$: fuzzy semiring $([0,1], \vee, \wedge, 0, 1)$

Examples

- for $a, b \in \mathbb{R}$ where $a < b$, the structure $([a, b], \vee, \otimes, a, b)$ where for all $x, y \in [a, b]$

$$x \vee y = \max\{x, y\} \quad x \otimes y = \max\{a, x + y - b\}$$

- $([0, 1], \max, \otimes, 0, 1)$
- $([0, 1] \cap \mathbb{Q}, \max, \otimes, 0, 1)$
- finite MV-semiring $\left(\left\{\frac{i}{n} \mid i \in \{0, \ldots, n\}\right\}, \max, \otimes, 0, 1\right)$
- finite MV-semiring $(\{0, \ldots, n\}, \max, \otimes, 0, n)$
- boolean MV-semiring $(\{0, 1\}, \max, \min, 0, 1)$
- every boolean algebra

Modified Łukasiewicz logic $\text{MSO}^{(\bar{Ł},W)}(A, \mathbb{X})$

fixed set of connectives:

- truth constants for every $c \in W$,
- connectives $\vee, \&$,
- $\neg$ only applied to atoms,
- quantifiers $\exists, \forall_{Ł}$.

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defined predicates:

$$
\begin{aligned}
\mathrm{first}(x) &= \forall_{Ł} y (x \leq y), \\
\mathrm{last}(x) &= \forall_{Ł} y (y \leq x) \\
S(x,y) &= (x \leq y) \,\&\, \neg(y \leq x) \,\&\, \forall_{Ł} z \, ((z \leq x) \vee (y \leq z))
\end{aligned}
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S(x,y) &= (x \leq y) \,\&\, \neg(y \leq x) \,\&\, \forall_{Ł} z\, ((z \leq x) \vee (y \leq z))
\end{aligned}$$

interpretation of formulas from $\mathrm{MSO}^{(\overline{Ł},W)}(A,\mathbb{X})$ use only truth functions $\vee$ (for $\vee$ and $\exists$) and $\otimes$ (for $\&$ and $\forall_{Ł}$)

for $\varphi \in \mathrm{MSO}^{(\overline{Ł},W)}(A,\mathbb{X})$, s_φ is a formal series

Weighted automata over MV-semirings

MV-semiring $\mathbb{W} = (W, \max, \otimes, 0_W, 1_W)$

$\mathbb{W}$-automaton $\mathcal{A} = (Q, \alpha, \delta, \beta)$ where

- $Q = \{1, \ldots, |Q|\}$ finite set of states
- $\alpha, \beta : Q \longrightarrow W$ initial and final vector
- $\delta : A^+ \longrightarrow (Q^2 \longrightarrow W)$ transition morphism
  uniquely defined by restriction $\delta : A \longrightarrow (Q^2 \longrightarrow W)$

behavior of $\mathcal{A}$: formal series $\|\mathcal{A}\| : A^* \longrightarrow W$ where

$$\|\mathcal{A}\|(w) = \max_{p_0, p_{|w|} \in Q} \left(\alpha(p_0) \otimes \delta(w)(p_0, p_{|w|}) \otimes \beta(p_{|w|})\right)$$

$$= \max_{p_0, p_{|w|} \in Q} \left(\alpha(p_0) \otimes \left(\max_{\substack{(p_0, \ldots, p_{|w|}) \\ \in Q^{|w|+1}}} \bigotimes_{i \in \{1, \ldots, |w|\}} \delta(w_i)\,(p_{i-1}, p_i) \right) \otimes \beta(p_{|w|}) \right)$$

formal series $S : A^* \longrightarrow \mathbb{W}$ is recognizable iff there is an automaton $\mathcal{A}$ such that $\|\mathcal{A}\| = S$

Weighted logic over MV-semirings

[Droste/Gastin: Weighted automata and weighted logics, ICALP'05]

MV-semiring $(W, \max, \otimes, 0_W, 1_W)$

syntax of MSO(W, A):

- **atoms** $x \leq y, P_a(x), x \in X$, where $x, y \in \mathbb{X}_1$, $a \in A$ and $X \in \mathbb{X}_2$
- **formulas** $\varphi ::= c \mid P \mid \neg P \mid \varphi * \psi \mid Qx\varphi$
  where $c \in W$, P atom, φ, ψ formulas, $* \in \{\vee, \wedge\}$, $Q \in \{\forall, \exists\}$ and $x \in \mathbb{X}$

similar to MSO$^{(\overline{Ł}, W)}(A, \mathbb{X})$

Weighted logic – semantics of atoms

for every pair (w, σ) where $w \in A^*$ and $\sigma : \mathbb{X} \longrightarrow 2^{\mathrm{pos}(w)}$:

semantic of truth values, atoms and negated atoms:

MSO(W, A)		MSO$^{(\bar{Ł}, W)}(A, \mathbb{X})$
$[\![c]\!]\,(w,\sigma) =$	c	$= [\![c]\!]_{(\underline{w},\sigma)}$
$[\![x \leq y]\!]\,(w,\sigma) =$	$\begin{cases} 1 & \text{iff } \sigma(x) \leq \sigma(y) \\ 0 & \text{otherwise} \end{cases}$	$= [\![x \leq y]\!]_{(\underline{w},\sigma)}$
$[\![P_a(x)]\!]\,(w,\sigma) =$	$\begin{cases} 1 & \text{iff } \sigma(x) > 0 \text{ and } w_{\sigma(x)} = a \\ 0 & \text{otherwise} \end{cases}$	$= [\![P_a(x)]\!]_{(\underline{w},\sigma)}$
$[\![x \in X]\!]\,(w,\sigma) =$	$\begin{cases} 1 & \text{iff } \sigma(x) \in \sigma(X) \\ 0 & \text{otherwise} \end{cases}$	$= [\![X(x)]\!]_{(\underline{w},\sigma)}$
$[\![\neg\varphi]\!]\,(w,\sigma) =$	$\begin{cases} 1 & \text{iff } [\![\varphi]\!]\,(w,\sigma) = 0 \\ 0 & \text{otherwise} \end{cases}$	$= [\![\neg\varphi]\!]_{(\underline{w},\sigma)}$

Weighted logic – semantics of formulas

for every pair (w, σ) where $w \in A^*$ and $\sigma : \mathbb{X} \longrightarrow 2^{\mathrm{pos}(w)}$:
MV-semiring $(W, \max, \otimes, 0_W, 1_W)$

semantic of non-atomic formulas

$\mathrm{MSO}(W, A)$ $\qquad\qquad$ $\mathrm{MSO}^{(\overline{Ł}, W)}(A, \mathbb{X})$

$$
\begin{aligned}
[\![\varphi \vee \psi]\!]\,(w,\sigma) &= \max\{[\![\varphi]\!]\,(w,\sigma), [\![\psi]\!]\,(w,\sigma)\} &&= [\![\varphi \vee \psi]\!]_{(\underline{w},\sigma)}\\
[\![\varphi \wedge \psi]\!]\,(w,\sigma) &= [\![\varphi]\!]\,(w,\sigma) \otimes [\![\psi]\!]\,(w,\sigma) &&= [\![\varphi \,\&\, \psi]\!]_{(\underline{w},\sigma)}\\
[\![\exists x \varphi]\!]\,(w,\sigma) &= \max_{i \in I} [\![\varphi]\!]\,(w, \sigma[x \mapsto i]) &&= [\![\exists x \varphi]\!]_{(\underline{w},\sigma)}\\
[\![\forall x \varphi]\!]\,(w,\sigma) &= \bigotimes_{i \in I} [\![\varphi]\!]\,(w, \sigma[x \mapsto i]) &&= [\![\forall_{Ł} x \varphi]\!]_{(\underline{w},\sigma)}
\end{aligned}
$$

where $I = \mathrm{pos}(w)$ for $x \in \mathbb{X}_1$ and $I = 2^{\mathrm{pos}(w)}$ for $x \in \mathbb{X}_2$

Translation t from $\mathrm{MSO}(W, A)$ to $\mathrm{MSO}^{(\overline{Ł}, W)}(A, \mathbb{X})$:

$$\wedge \mapsto \& \qquad \forall \mapsto \forall_{Ł}$$

Connection between $\mathrm{MSO}(W, A)$ and $\mathrm{MSO}^{(\bar{Ł}, W)}(A, \mathbb{X})$

for sentence $\varphi \in \mathrm{MSO}(W, A)$: $\quad [\![\varphi]\!] : A^* \longrightarrow W$

$\varphi \in \mathrm{MSO}^{(\bar{Ł}, W)}(A, \mathbb{X})$: $\quad s_\varphi : A^* \longrightarrow W$

Proposition

For every sentence $\varphi \in \mathrm{MSO}(W, A)$*:* $[\![\varphi]\!] = s_{t(\varphi)}$

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For every sentence $\varphi \in \mathrm{MSO}(W, A)$*:* $[\![\varphi]\!] = s_{t(\varphi)}$

formal series $S : A^* \longrightarrow W$ is definable in

$\mathrm{MSO}(W, A)$	iff there is a $\varphi \in \mathrm{MSO}(W, A)$ such that $[\![\varphi]\!] = S$.
$\mathrm{MSO}^{(\bar{\text{Ł}}, W)}(A, \mathbb{X})$	iff there is a $\varphi \in \mathrm{MSO}^{(\bar{\text{Ł}}, W)}(W, A)$ such that $s_\varphi = S$.

Proposition

A formal series $S : A^* \longrightarrow W$ *is definable in* $\mathrm{MSO}(W, A)$ *iff* S *is definable in* $\mathrm{MSO}^{(\bar{\text{Ł}}, W)}(A, \mathbb{X})$*.*

Recognizability vs. $\mathrm{MSO}^{(\overline{Ł},W)}(A,\mathbb{X})$-definability

Theorem (Droste/Gastin 2005)

Let $\mathbb{K}$ be a locally finite semiring and A an alphabet. Then a series $S : A^* \longrightarrow K$ is recognizable iff S is $\mathrm{MSO}(W, A)$*-definable.*

MV-semiring $\mathbb{W} = (W, \max, \otimes, 0_W, 1_W)$ is commutative, locally finite

series S is $\mathrm{MSO}(W, A)$-definable iff
S is $\mathrm{MSO}^{(\overline{Ł},W)}(A,\mathbb{X})$-definable

Corollary

For any MV-semiring $\mathbb{W}$ and alphabet A, a series $S : A^* \longrightarrow K$ is recognizable iff S is $\mathrm{MSO}^{(\overline{Ł},W)}(A,\mathbb{X})$*-definable.*

$\mathrm{MSO}^{(\bar{Ł},W)}(A,\mathbb{X})$-formula for $\mathbb{W}$-automaton

$\mathbb{W}$-automaton $\mathcal{A} = (Q, \alpha, \delta, \beta)$

$$\varphi_{\mathcal{A}} = \exists X_1 \ldots \exists X_{|Q|} \, (\varphi_p \,\&\, \varphi_\alpha \,\&\, \varphi_\delta \,\&\, \varphi_\beta) \qquad \text{where}$$

$$\varphi_p = \forall_{Ł} x \bigvee_{p \in Q} \left(X_p(x) \,\&\, \mathop{\&}_{q \in Q \setminus \{p\}} \neg X_q(x) \right)$$

$$\varphi_\alpha = \forall_{Ł} x \left(\neg \mathrm{first}(x) \vee \bigvee_{p \in Q} (X_p(x) \,\&\, \alpha(p)) \right)$$

$$\varphi_\delta = \forall_{Ł} x \forall_{Ł} y \left(\begin{array}{l} \neg S(x,y) \vee \\ \displaystyle\bigvee_{\substack{p \neq q \in Q \\ a \in A}} (X_p(x) \,\&\, X_q(y) \,\&\, P_a(y) \,\&\, \delta(a)(p,q)) \end{array} \right)$$

$$\varphi_\beta = \forall_{Ł} x \left(\neg \mathrm{last}(x) \vee \bigvee_{p \in Q} (X_p(x) \,\&\, \beta(p)) \right)$$

Decidability results

Theorem (Droste/Gastin 2005)

For a locally finite commutative semiring $\mathbb{K}$ *and an alphabet* A*, it is decidable*

1. *for* $\varphi, \psi \in \mathrm{MSO}(\mathbb{K}, A)$*, whether* $[\![\varphi]\!] = [\![\psi]\!]$
2. *for* $\varphi \in \mathrm{MSO}(\mathbb{K}, A)$*, whether* $0 \in [\![\varphi]\!]\,(A^*)$

Corollary

For an MV-semiring $\mathbb{W}$ *and an alphabet* A*, it is decidable*

1. *for* $\varphi, \psi \in \mathrm{MSO}^{(\overline{Ł}, W)}(A, \mathbb{X})$*, whether* $s_\varphi = s_\psi$
2. *for* $\varphi \in \mathrm{MSO}^{(\overline{Ł}, W)}(A, \mathbb{X})$*, whether* $0 \in s_\varphi\,(A^*)$

Summary

logic	Łukasiewicz	weighted

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truth domain	MV-algebras, locally finite semirings	semirings
connectives	lattice operations $\vee, \wedge$, new connectives $\&, \underline{\vee}$	$\vee, \wedge$ semiring operations

results on recognizability and decidability for locally finite commutative semirings and weigted logic
applied to MV-semirings and $\mathrm{MSO}^{(Ł,w)}(A, \mathbb{X})$