Recognizability of Iterative Picture Languages

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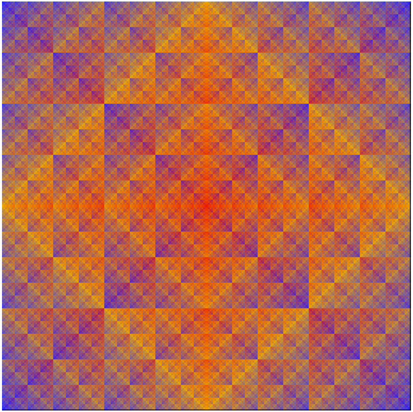
Motivation

Fractal image generation / compression:
iterated generation of finite languages over a finite set C of colors ($[0,1]^3$ RGB)



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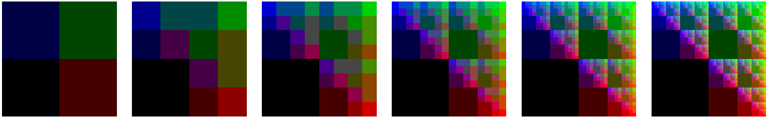

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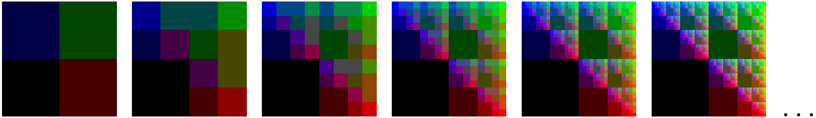

positions in real unit interval (partition into subsquares)

Sequence of stepwise generated pictures as picture language over a finite set of colors C

Problem: Are these picture languages recognizable?

Pictures and Picture Languages

finite alphabet C of Colors

$$p : \{0, \ldots, m-1\} \times \{0, \ldots, n-1\} \to C$$

is a C-picture of size $m \times n$

$$\mathrm{dom}(p) = \{0, \ldots, m-1\} \times \{0, \ldots, n-1\}$$

Set of all C-pictures of size $m \times n$: $C^{m \times n}$

Set of all finite C-pictures: $C^{++} = \bigcup_{m,n \in \mathbb{N}_+} C^{m \times n}$

$P \subseteq C^{++}$ is called picture language

Subpictures

$q \in C^{m' \times n'}$ is a $m' \times n'$ subpicture of $p \in C^{m \times n}$ if there is a position $(i, j) \in \text{dom}(p)$ such that for all $(i', j') \in \text{dom}(q)$:

$$p(i + i', j + j') = q(i', j')$$

Example

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$B_{2,2}(p)$	Set of all 2×2 subpictures of the picture $p \in C^{++}$
$B_{2,2}(P)$	$= \bigcup_{p \in P} B_{2,2}(p)$ for $P \subseteq C^{++}$

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$B_{2,2}$ and $E_{2,2}$ are not mutually inverse.

Recognizable Picture Languages

picture $p \in C^{m \times n}$

$$p = \begin{array}{|cccc|} \hline 1 & 2 & 2 & 3 \\ 1 & 2 & 2 & 3 \\ 1 & 2 & 2 & 3 \\ 1 & 2 & 2 & 3 \\ \hline \end{array}$$

Recognizable Picture Languages

picture $p \in C^{m \times n}$

framed Picture $\hat{p} \in (C \cup \{\#\})^{(m+2)\times(n+2)}$

$$\hat{p} = \begin{array}{|cccccc|} \hline \# & \# & \# & \# & \# & \# \\ \# & 1 & 2 & 2 & 3 & \# \\ \# & 1 & 2 & 2 & 3 & \# \\ \# & 1 & 2 & 2 & 3 & \# \\ \# & 1 & 2 & 2 & 3 & \# \\ \# & \# & \# & \# & \# & \# \\ \hline \end{array}$$

Recognizable Picture Languages

picture $p \in C^{m \times n}$
framed Picture $\hat{p} \in (C \cup \{\#\})^{(m+2)\times(n+2)}$
$P \subseteq C^{++}$: $\hat{P} = \{\hat{p} \mid p \in P\}$

Picture language $P \subseteq C^{++}$ is local iff

$$\hat{P} = E_{2,2}(B_{2,2}(\hat{P}))$$

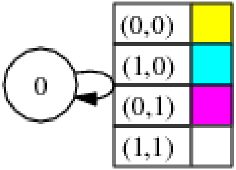
Picture language $P \subseteq C^{++}$ is recognizable iff there is a local picture language $P' \subseteq D^{++}$ and a morphism $\varphi : D \to C$ such that $P = \{\varphi(p) \mid p \in P'\}$ (pointwise application of φ to p).

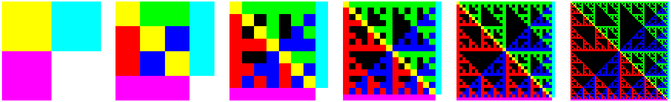
Theorem (Giammaresi, Restivo)

The set of all recognizable picture languages is closed under intersection and projection.

Examples

1. colored Sierpinski triangles
   weight semiring $([0,1]^3, \max, \min, 0, 1)$





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2. Set of all monochrome squares of color $c \in C$ with side 2^n for $n \in \mathbb{N}$ is not local but recognizable.

   local language S,
   projektion $\varphi : C \longrightarrow C$ where for every $d \in C$: $\varphi(d) = c$

Finite words and positions

(Kari / Čulik)

alphabet $A_2 = \{0, 1\}^2$

projektions $\pi_1, \pi_2 : A_2 \longrightarrow \{0, 1\}$

bijektion $\text{Pos} : A_2^n \longrightarrow \{0, \ldots, 2^n - 1\}^2$ where for all $w \in A_2^n$:

$$\text{Pos}(w) = \left(\sum_{i=0}^{|w|-1} \pi_1(w_i) 2^{|w|-i-1}, \sum_{i=0}^{|w|-1} \pi_2(w_i) 2^{|w|-i-1} \right)$$

Every word $w \in A_2^n$ defines a position in $\{0, \ldots, 2^n - 1\}^2$.

positions in quadrants

Inverse function Pos^{-1} maps positions to words over A_2.

Picture languages from weighted word languages

Every weighted word language $L : A_2^+ \to \mathbb{W}$ that assumes only finitely many values from $\mathbb{W}$ defines a picture language

$$\text{picture}(L) = \{p_i \mid i \in \mathbb{N}_+\}$$

of pictures with colors from a finite subset of $\mathbb{W}$ where for every $p_i : \{0, \ldots, 2^i - 1\}^2 \to \{0, 1\}$ and every $(m, n) \in \{0, \ldots, 2^i - 1\}^2$:

$$p_i(m, n) = L(\text{Pos}^{-1}(m, n))$$

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Example: $L_S = ((0, 0) + (1, 1) + (1, 0))^*$ defines the picure language S of refinements of (black and white) Sierpinski triangles.

Recognizable word languages over locally finite semirings

semiring $\mathbb{W} = (W, +_W, \cdot_W, 0_W, 1_W)$ is locally finite iff every finitely generated set $U \subseteq W$ is finite.

Examples: $(\{0,1\}, \max, \min, 0, 1)$, $([0,1], \max, \min, 0, 1)$, $([0,1], \min, \oplus, 0, 1)$ where $x \oplus y = \min(1, x+y)$, $([0,1], \max, \odot, 0, 1)$ where $x \odot y = \max(0, 1-x-y)$

Theorem (Droste, Gastin 2000)

Every recognizable word language $L : A_k^* \to W$ over a locally finite semiring $\mathbb{W}$ is representable as finite sum

$$L = \sum_{i=1}^{n} c_i L_i$$

where for every $i \in \{1, \dots, n\}$: $c_i \in W$ and L_i is a classical regular word language (values from $\{0_W, 1_W\}$).

The languages L_i can be assumed to be disjoint.

Problems

1. Is for every recognizable weighted word language $L : A_k^* \to \mathbb{W}$ over a locally finite semiring $\mathbb{W}$ the picture language picture(L) recognizable?

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2. Is for every recognizable picture language picture(L) the word language L recognizable?

Two dimensional D0L systems

D0L system $\mathcal{S} = (A, R, a)$ with alphabet A, Axiom $a \in A$ and rules $R : A \to A^{2\times 2}$.

Example:

$$\mathcal{S} = (\{0, 1\}, R, 1) \quad \text{where} \quad R = \left\{ 0 \to \boxed{\begin{matrix} 0 & 0 \\ 0 & 0 \end{matrix}}, 1 \to \boxed{\begin{matrix} 1 & 0 \\ 1 & 1 \end{matrix}} \right\}$$

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iterated application of R to a defines a picture language

$$\text{picture}(\mathcal{S}) = \left\{ q_i \in A^{2^i \times 2^i} \mid i \in \mathbb{N}_+ \right\} \qquad \text{by}$$

$$\begin{aligned} q_1 &= R(a) \\ \text{for all } i \in \mathbb{N}_+ : \quad q_{i+1} &= R(q_i) \quad \text{(pointwise application)} \end{aligned}$$

Example

$$\mathcal{S} = (\{0,1\}, R, 1) \quad \text{where} \quad R = \left\{ 0 \to \begin{array}{|cc|}\hline 0 & 0 \\ 0 & 0 \\ \hline\end{array}, 1 \to \begin{array}{|cc|}\hline 1 & 0 \\ 1 & 1 \\ \hline\end{array} \right\}$$

$$\text{picture}(\mathcal{S}) =$$

$$\left\{ \begin{array}{|cc|}\hline 1 & 0 \\ 1 & 1 \\ \hline\end{array}, \begin{array}{|cc|cc|}\hline 1 & 0 & 0 & 0 \\ 1 & 1 & 0 & 0 \\ \hline 1 & 0 & 1 & 0 \\ 1 & 1 & 1 & 1 \\ \hline\end{array}, \begin{array}{|cc|cc|cc|cc|}\hline 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 1 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ \hline 1 & 0 & 1 & 0 & 0 & 0 & 0 & 0 \\ 1 & 1 & 1 & 1 & 0 & 0 & 0 & 0 \\ \hline 1 & 0 & 1 & 0 & 1 & 0 & 0 & 0 \\ 1 & 1 & 1 & 1 & 1 & 1 & 0 & 0 \\ \hline 1 & 0 & 1 & 0 & 1 & 0 & 1 & 0 \\ 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 \\ \hline\end{array}, \ldots \right\}$$

D0L picture languages are recognizable

Satz (Lindgren, Moore, Nordahl 2000)

Every picture language generated by a D0L system is recognizable.

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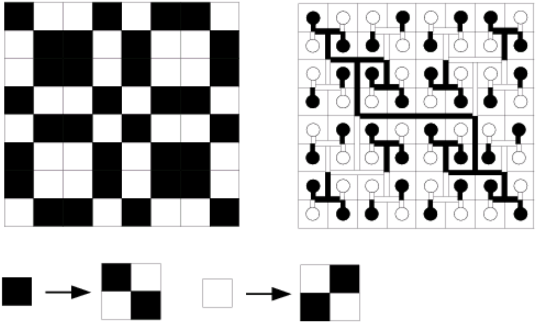
Proof sketch:
additional information structure guarantees hierarchic structure of the pictures
(ideas from tilings of the infinite plane, Wang, Robinson, 1960)



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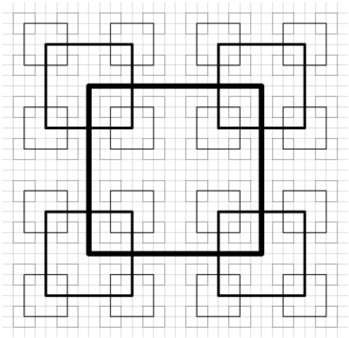

$\mathbb{W}$-DFA over locally finite semirings $\mathbb{W}$

recognizable word language $L : A_k^* \to W$ over a locally finite semiring $\mathbb{W}$

$$L = \sum_{i=1}^{n} c_i L_i$$

$c_i \in W$ and all L_i classically recognizable and mutually disjoint

Every L_i is recognized by a complete DFA $\mathcal{A}_i = (Q_i, \delta_i, s_i, F_i)$

product automaton ($\mathbb{W}$-DFA) $\mathcal{A} = (\times_{i\in\{1,\dots,n\}} Q_i, \delta, \alpha, \beta)$ where for every $a \in X$:

$$\delta(a)((p_1,\dots,p_n),(q_1,\dots,q_n)) = \begin{cases} 1_W & \text{iff } \forall i \in \{1,\dots,n\} : (p_i,q_i) \in \delta_i(a) \\ 0_W & \text{otherwise} \end{cases}$$

$$\alpha(p_1,\dots,p_n) = \begin{cases} 1_W & \text{iff } \forall i \in \{1,\dots,n\} : p_i = s_i \\ 0_W & \text{otherwise} \end{cases}$$

$$\beta(p_1,\dots,p_n) = \begin{cases} c_i & \text{iff } p_i \in F_i \\ 0_W & \text{otherwise} \end{cases}$$

$\mathbb{W}$-DFA over locally finite semirings $\mathbb{W}$

Properties of the product automaton $\mathcal{A}$:

- complete and deterministic (by construction)
- Every state $(p_1, \ldots, p_n)$ reachable from $(s_1, \ldots, s_n)$ contains at most one $i \in \{1, \ldots, n\}$ accepting state $p_i \in F_i$ (since all L_i are mutually disjoint).
- $L(\mathcal{A}) = L$

Picture languages from $\mathbb{W}$-DFA

Every $\mathbb{W}$-DFA $\mathcal{A} = (A_2, Q, \delta, \alpha, \beta)$ defines a picture language

$$\text{picture}(\mathcal{A}) = \left\{ p_i \in Q^{2^i \times 2^i} \mid i \in \mathbb{N}_+ \right\},$$

where for all $(m, n) \in \{0, \ldots, 2^i - 1\}^2$:

$$p_i(m, n) = \delta\left(s, \text{Pos}^{-1}(m, n)\right)$$

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pointwise application of $\beta : Q \longrightarrow \mathbb{W}$ to a picture $q \in Q^{2^i \times 2^i}$ with colors from Q generates a picture of the same size with colors from $\mathbb{W}$ by $\beta(q) = c \in \mathbb{W}^{2^i \times 2^i}$.

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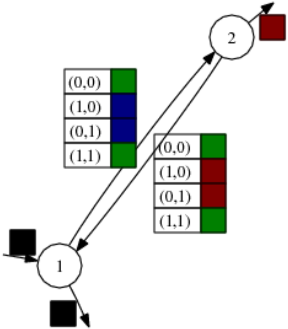
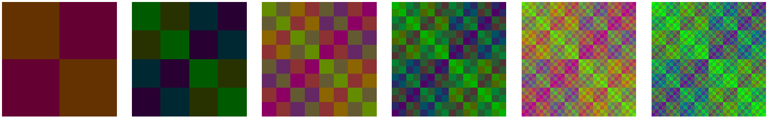
Proposition

Then $\beta\,(\text{picture}(\mathcal{A})) = \text{picture}(L(\mathcal{A}))$
(pointwise application of β *to every picture in* $\text{picture}(\mathcal{A})$*).*

Example

semiring $([0,1]^3, \min, \oplus, 1, 0)$





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$\mathbb{W}$-DFA and D0L systems

Every $\mathbb{W}$-DFA $\mathcal{A} = (A_2, Q, s, \delta, F)$ over a locally finite semiring $\mathbb{W}$ defines a D0L system $\mathcal{S}_{\mathcal{A}} = (Q, R, s)$ where

$$R = \left\{ q \longrightarrow \boxed{\begin{matrix} \delta(q,(0,0)) & \delta(q,(0,1)) \\ \delta(q,(1,0)) & \delta(q,(1,1)) \end{matrix}} \mid q \in Q \right\}$$

and picture$(\mathcal{A})$ = picture$(\mathcal{S}_{\mathcal{A}})$.

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and $\text{picture}(\mathcal{A}) = \text{picture}\,(\mathcal{S}_{\mathcal{A}})$.

Hence the word language L recognized by $\mathcal{A}$ satisfies

Lemma

$\text{picture}(L) = \beta\,(\text{picture}\,(\mathcal{S}_{\mathcal{A}}))$

Problem 1

Theorem (Lindgren / Moore / Nordahl, 2000)

Every picture language generated by a D0L system is recognizable.

Corollary

Every picture language generated by a recognizable word language over a locally finite semiring is recognizable.

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Proof:
given recognizable word language $L : A_k^* \to \mathbb{W}$
construct $\mathbb{W}$-WFA $\mathcal{A}$ that recognizes L,
construct D0L system $\mathcal{S}_{\mathcal{A}}$ for $\mathcal{A}$,
picture($\mathcal{S}_{\mathcal{A}}$) is recognized by (P, φ) (Theorem),
picture(L) is recognized by $(P, \varphi \circ \beta)$

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Is for every recognizable picture language picture(L) the word language L recognizable?

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L is not a recognizable word language but picture(L) is a recognizable picture language.

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$M_1 = \{2^{2^n} \mid n \in \mathbb{N}\}$, $M_2 = \{2^n \mid n \in \mathbb{N}\} \setminus M_1$

$P_1 = \{p_i \in \{1\}^{m \times m} \mid m \in M_1\}$

(black squares of side 2^{2^n})

$P_2 = \{p_i \in \{0\}^{m \times m} \mid m \in M_2\}$

(white squares of side 2^i where $i \notin \{2^n \mid n \in \mathbb{N}\}$)

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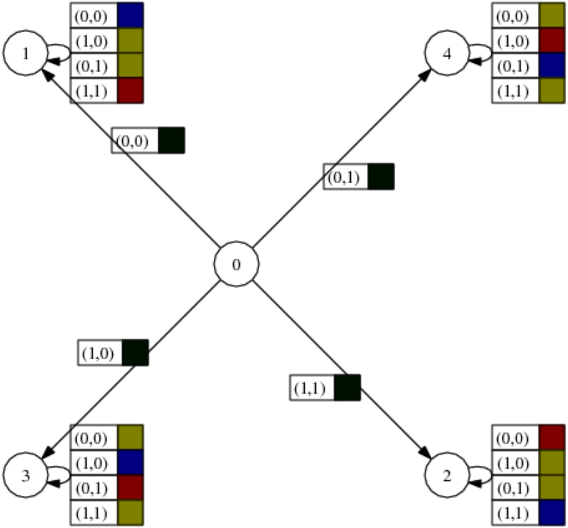
Satz (Kari / Moore, 2004)

For every recursively enumerable set $M \subseteq \mathbb{N}$, the set of all white squares with side $s \in M$ is recognizable.

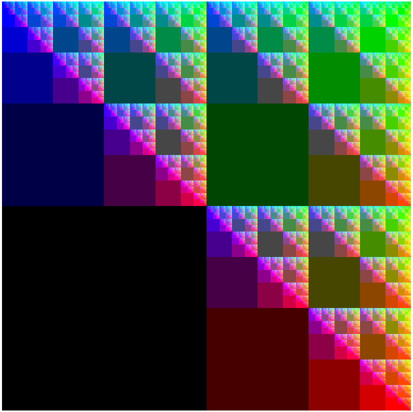
$M_1 = \{2^{2^n} \mid n \in \mathbb{N}\}$ and $M_2 = \{2^n \mid n \in \mathbb{N}\} \setminus M_1$ are recursively enumerable.

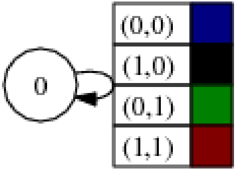
P_1 und P_2 is recognizable (Theorem).

Then picture(L) $= P_1 \cup P_2$ is recognizable.



weight semiring $([0,1]^3, \min, \oplus, 1, 0)$




0
(0,0)
(1,0)
(0,1)
(1,1)

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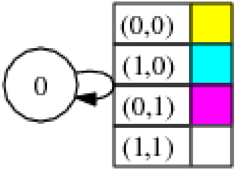

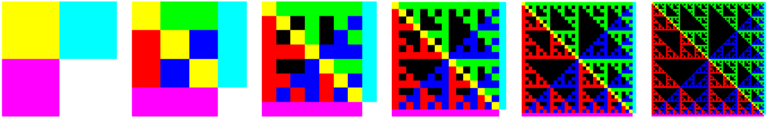

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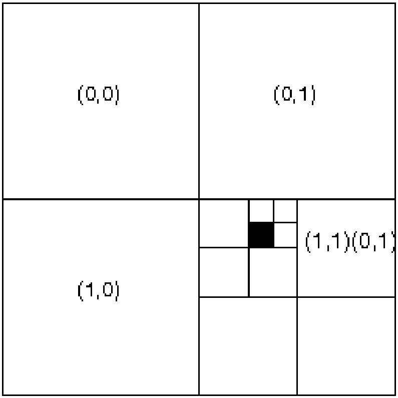
Iterated division of $[0, 1]^2$ into subsqares



Position of the back square:
$(1, 1)(0, 0)(0, 1)(1, 0)$

Example

$C = \{1, 2, 3\}$

$$\boxed{\begin{matrix} 1 & 2 \\ 1 & 2 \end{matrix}}, \quad \boxed{\begin{matrix} 2 & 2 \\ 2 & 2 \end{matrix}}, \quad \boxed{\begin{matrix} 2 & 3 \\ 2 & 3 \end{matrix}}$$

are 2×2 subpictures of

$$p = \boxed{\begin{matrix} 1 & 2 & 2 & 3 \\ 1 & 2 & 2 & 3 \\ 1 & 2 & 2 & 3 \\ 1 & 2 & 2 & 3 \end{matrix}}$$

Example

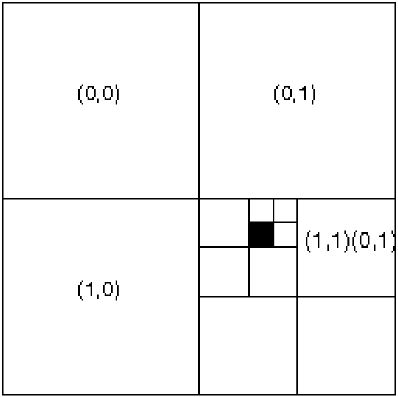
$$E_{2,2}\left(\left\{\boxed{\begin{matrix}1 & 2\\ 1 & 2\end{matrix}},\ \boxed{\begin{matrix}2 & 2\\ 2 & 2\end{matrix}},\ \boxed{\begin{matrix}2 & 3\\ 2 & 3\end{matrix}}\right\}\right)$$

contains every picture of the following shape

$$\boxed{\begin{matrix}1 & 2 & \dots & 2 & 3\\ \vdots & \vdots & \ddots & \vdots & \vdots\\ 1 & 2 & \dots & 2 & 3\end{matrix}} \quad \boxed{\begin{matrix}2 & 2 & \dots & 2 & 3\\ \vdots & \vdots & \ddots & \vdots & \vdots\\ 2 & 2 & \dots & 2 & 3\end{matrix}},$$

$$\boxed{\begin{matrix}1 & 2 & \dots & 2 & 2\\ \vdots & \vdots & \ddots & \vdots & \vdots\\ 1 & 2 & \dots & 2 & 2\end{matrix}} \quad \boxed{\begin{matrix}2 & 2 & \dots & 2 & 2\\ \vdots & \vdots & \ddots & \vdots & \vdots\\ 2 & 2 & \dots & 2 & 2\end{matrix}}$$

Iterated division of $[0, 1]^2$ into subsquares



Position of the black square:
$(1, 1)(0, 0)(0, 1)(1, 0)$
Position of the black square in $\{0, \ldots, 2^4 - 1\}^2$:
$(2^3 + 2^0, 2^3 + 2^1) = (9, 10)$