Recognizability of Iterative Picture Languages

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1 Motivation

Pictures are rectangular arrays of colors from a finite alphabet. A set of pictures is called *picture language*. Picture languages are studied in various contexts such as pattern recognition and digital image processing [7].

A special class of picture languages can be defined by word languages over an alphabet $A_k = \{0, \ldots, k-1\}^2$ of coordinates [7]. We call such languages *iterative* picture languages. Iterative picture languages that are generated by regular word languages are used, for example, in digital image compression [7] and generation of fractal pictures [2].

We show that every iterative picture language generated by a regular word language is recognizable by a tiling system [1]. Furthermore, we present a non-regular word language that generates a recognizable picture language.

2 Iterative picture languages

Some languages of black and white pictures can be defined by word languages. We call such languages *iterative* picture languages. Every word $w = w_1 \cdots w_n$ over an alphabet $A_k = \{0, \ldots, k-1\}^2$ addresses a position $\mathsf{Pos}(w) \in \{0, \ldots, k^n - 1\}^2$ by

$$\mathsf{Pos}(w) = \left(\sum_{i=0}^{|w|-1} \pi_1(w_i) k^{|w|-i-1}, \sum_{i=0}^{|w|-1} \pi_2(w_i) k^{|w|-i-1} \right)$$

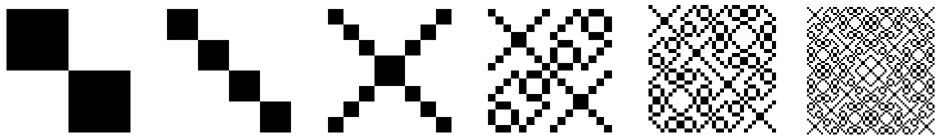
where $\pi_1(w)$ and $\pi_2(w)$ are the projections of letters $(m, n) \in \{0, \ldots, k-1\}^2$ to its first and second component. The inverse Pos^{-1} maps every position in a $k^n \times k^n$-square to a word in A_k^n. Every word language $L \subseteq A_k^+$ defines a set $\mathsf{picture}(L) = \{p_i \mid i \in \mathbb{N}_+\}$ of black and white pictures where

$$p_i(m,n) = \begin{cases} 1 & \text{if } \mathsf{Pos}^{-1}(m,n) \in L \\ 0 & \text{otherwise} \end{cases}$$

For example, the regular word language

$$L = (((0,0) + (1,1))^*((0,1) + (1,0)))^{3*}((0,0) + (1,1))^*$$

defines the picture language $\mathsf{picture}(L) = \{p_i \mid i \in \mathbb{N}_+\}$ where



p_1 p_2 p_3 p_4 p_5 p_6

By definition, every iterative picture language P generated by a word language $L \subseteq A_k^+$ satisfies the following property:

(PS) for every $i \in \mathbb{N}_+$, P contains exactly one picture $p_i : \{0, \ldots, k^i - 1\}^2 \to \{0, 1\}$.

3 Recognizable picture languages

Recognizability of picture languages is defined by tiling systems [1]. This definition coincides with recognizability of picture languages by several other computational devices (nondeterministic 4-way-automata, on-line tessellation automata).

For a picture language P, the set $B_{2,2}(P)$ contains all 2×2-sub-pictures of pictures in P. Given a set T of 2×2-pictures, the set $E_{2,2}(T)$ contains all pictures p satisfying $B_{2,2}(\{p\}) \subseteq T$. Note that in general, $B_{2,2}$ and $E_{2,2}$ are not inverse to each other.

For every picture $p : \{0, \ldots, m\} \times \{0, \ldots, m\} \to C$, the framed picture $\hat{p} : \{0, \ldots, m+2\} \times \{0, \ldots, m+2\} \to C \cup \{\#\}$ is defined by

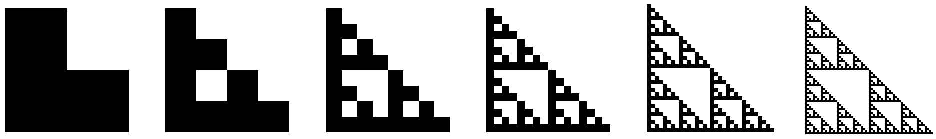
$$\hat{p}(i,j) = \begin{cases} p(i-1, j-1) & \text{if } (i-1, j-1) \in \{0, \ldots, m\} \times \{0, \ldots, n\} \\ \# & \text{otherwise} \end{cases}$$

This is extended to picture languages by $\hat{P} = \{\hat{p} \mid p \in P\}$.

A picture language P is called *local* if $\hat{P} = E_{2,2}(B_{2,2}(\hat{P}))$. A picture language P is called *recognizable* if P is a homomorphic image of a local picture language.

For example, the language P_S of finite Sierpinski triangles
$P_S = \{p_i : \{0, \ldots, 2^i - 1\}^2 \to \{0,1\} \mid i \in \mathbb{N}_+\}$ where for all $i \in \mathbb{N}_+$ and

$$k \in \{0, \ldots, 2^i - 1\} : p_i(k, 0) = 1$$
$$l \in \{1, \ldots, 2^i - 1\} : p_i(0, l) = 0$$
$$(k,l) \in \{1, \ldots, 2^i - 1\}^2 : p_i(k,l) = \begin{cases} 1 & \text{if } p_i(k-1, l-1) + p_i(k-1, l) = 1 \mod 2 \\ 0 & \text{otherwise} \end{cases}$$



p_1 p_2 p_3 p_4 p_5 p_6

is local [4].

The set P_2 of all monochrome white squares of side 2^n where $n \in \mathbb{N}_+$ is not local but recognized by the local language P_S defined above and the mapping $\varphi(0) = \varphi(1) = 0$.

4 Picture languages generated by regular languages

Picture languages defined by regular languages are used for example in digital image compression [7] and generation of fractal pictures [2]. Every regular word language $L \subseteq A^*$ is recognized by a complete deterministic finite automaton $\mathcal{A} = (A, Q, \delta, s, F)$. By stepwise computation, this automaton defines a language $\mathsf{picture}(\mathcal{A}) = \{p_i \mid i \in \mathbb{N}_+\}$ of pictures with colors from Q by

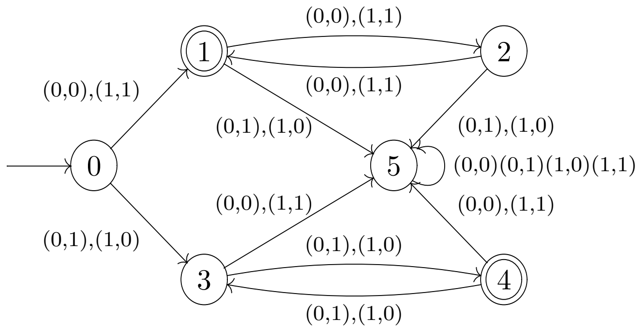
$$p_i(j, l) = \delta(s, \mathsf{Pos}^{-1}(j, l)).$$

Every complete deterministic finite automaton $\mathcal{A} = (A, Q, \delta, s, F)$ defines a two-dimensional D0L-System (Q, R, s) where

$$\text{for every state } q \in Q: \quad R(q) = \boxed{\begin{matrix} \delta(q,(0,0)) & \cdots & \delta(q,(0,k-1)) \\ \vdots & \ddots & \vdots \\ \delta(q,(k-1,0)) & \cdots & \delta(q,(k-1,k-1)) \end{matrix}}$$

For every D0L-system, iterated simultaneous application of the substitution of $q \in Q$ by $R(q)$ generates a picture language. The D0L-system defined above generates the picture language $\mathsf{picture}(\mathcal{A})$.

For example, the automaton



generates the picture language picture$(\mathcal{A}) = \{p_i : \{0, \dots, 2^i - 1\}^2 \to \{0, \dots, 5\} \mid i \in \mathbb{N}_+\}$ where for all $j \in \mathbb{N}_+$

$$p_{2j-1} = \begin{array}{|ccccccc|}\hline \mathbf{1} & 5 & 5 & \cdots & 5 & 5 & 3 \\ 5 & \mathbf{1} & 5 & \cdots & 5 & 3 & 5 \\ 5 & 5 & \mathbf{1} & \cdots & 3 & 5 & 5 \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots & \vdots \\ 5 & 5 & 3 & \cdots & \mathbf{1} & 5 & 5 \\ 5 & 3 & 5 & \cdots & 5 & \mathbf{1} & 5 \\ 3 & 5 & 5 & \cdots & 5 & 5 & \mathbf{1} \\ \hline \end{array} \quad \text{and } p_{2j} = \begin{array}{|ccccccc|}\hline 2 & 5 & 5 & \cdots & 5 & 5 & \mathbf{4} \\ 5 & 2 & 5 & \cdots & 5 & \mathbf{4} & 5 \\ 5 & 5 & 2 & \cdots & \mathbf{4} & 5 & 5 \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots & \vdots \\ 5 & 5 & \mathbf{4} & \cdots & 2 & 5 & 5 \\ 5 & \mathbf{4} & 5 & \cdots & 5 & 2 & 5 \\ \mathbf{4} & 5 & 5 & \cdots & 5 & 5 & 2 \\ \hline \end{array}$$

Application of the characteristic function of the set of final states results in a language of white squares with black diagonals in alternating direction.

In [4], it was stated (with a proof sketch) that every picture language generated by a two-dimensional D0L-system is recognizable. The idea of the proof relies on recognizability of an auxiliary information structure [6]. A similar structure is used in [5].

Pointwise application of the characteristic function $\chi_F : Q \to \{0, 1\}$ of the set F of final states of the automaton maps every picture $p_i : \{0, \dots, k^i - 1\}^2 \to Q$ to a picture $q_i : \{0, \dots, k^i - 1\}^2 \to \{0, 1\}$. Hence we have picture$(L) = \chi_F($picture$(\mathcal{A}))$ and get the following result.

Theorem 1 *For every regular word language $L \subseteq A_k^+$, the picture language* picture(L) *is recognizable.*

Since the property (PS) defined in the end of Section 2 is a strong restriction to a picture language, one might conjecture that every recognizable picture language satisfying this property (PS) is picture(L) for a regular word language L. The following picture language is a counterexample.

The word language $L = \{A_2^{2^n} \mid n \in \mathbb{N}_+\}$ is not regular by a simple pumping argument. $\mathsf{picture}(L)$ is a set of monochrome pictures p_i. p_i is a black square of side 2^i if $i = 2^n$ for an $n \in \mathbb{N}_+$ and otherwise a white square.

Kari and Moore showed in [3] that for every recursively enumerable set M of natural numbers, the set of all monochrome squares of side $m \in M$ is recognizable. Both $M_1 = \{2^{2^n} \mid n \in \mathbb{N}\}$ and $M_2 = \{2^n \mid n \in \mathbb{N}\} \setminus M_1$ are recursively enumerable. P_1 is the set of all black squares of side from M_1 and P_2 is the set of all white squares of side from M_2. By the result from [3], P_1 and P_2 are recognizable and so is their union $\mathsf{picture}(L) = P_1 \cup P_2$.

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