ω-automata: Topology and Measure

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Quantitative Logics and Automata, Leipzig
December 1, 2015

Outline

ω-automata

- Circuit Design
  - Monadic Second-Order Logic
- Verification
  - Temporal Logics
  - Fixed-Point Logics
  - Model Checking
- Symbolic Dynamics

Model Checking: Automata-theoretic properties

Quotation from a recent paper by DIEKERT, MUSCHOLL and WALUKIEWICZ

The common theme in *automata on infinite words* is that finite state devices serve to classify ω-regular properties. The most prominent classes are:

Deterministic properties: there exists a DBA.[a]

Deterministic properties which are simultaneously co-deterministic: there exists a DWA.

Safety properties: there exists a DBA where all states are final.

Cosafety properties: the complement is a safety property.

Liveness properties: there exists a BA where from all states there is a path to some final state lying in a strongly connected component.

Monitorable properties: there exists a monitor."

[a] deterministic BÜCHI automaton

Relations to Topology

Correspondence to topological properties

Safety: closed sets $= \mathbb{F}$.

Co-safety: open sets $= \mathbb{G}$.

Liveness: dense = closure is the whole space.

Deterministic: $\mathbb{G}_\delta$

Co-deterministic: $\mathbb{F}_\sigma$

Deterministic and simultaneously co-deterministic: $\mathbb{G}_\delta \cap \mathbb{F}_\sigma$

Monitorable: the boundary is *nowhere dense*.

$\Longrightarrow$

Fair Correctness [Varacca and Völzer]

The set of runs which satisfy the specification is *large* from a topological point of view.

Notation: Strings and Languages

Finite Alphabet $X = \{0, \dots, r-1\}$, cardinality $|X| = r$

Finite strings (words) $w = x_1 \cdots x_n \in \{0,1\}^*$, $x_i \in \{0,1\}$

Length $|w| = n$

Languages $W \subseteq X^*$

Infinite strings (ω-words) $\xi = x_1 \cdots x_n \cdots \in X^\omega$

Prefixes of infinite strings $\xi \upharpoonright n \in X^*$, $|\xi \upharpoonright n| = n$

$$\mathbf{pref}(\xi) = \{\xi \upharpoonright n : n \in \mathbb{N}\}$$

ω-Languages $F \subseteq X^\omega$

X^ω as CANTOR space

Metric: $\rho(\eta,\xi) := \inf\{r^{-|w|} : w \in \mathbf{pref}(\eta) \cap \mathbf{pref}(\xi)\}$

Balls: $w \cdot X^\omega = \{\eta : w \in \mathbf{pref}(\eta)\} = \{\eta : w \sqsubset \eta\}$

Diameter: $\operatorname{diam} w \cdot X^\omega = r^{-|w|}$

$\operatorname{diam} F = \inf\{r^{-|w|} : F \subseteq w \cdot X^\omega\}$

Open sets: $W \cdot X^\omega = \bigcup_{w \in W} w \cdot X^\omega$

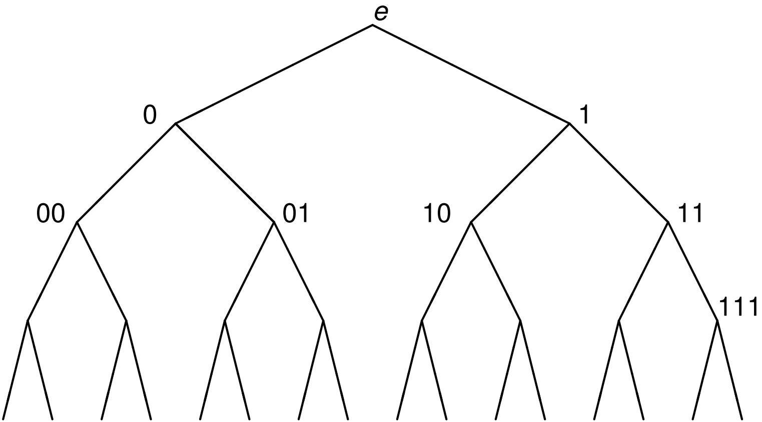
Closure: (Smallest closed set containing F)

$\mathcal{C}(F) = \{\xi : \mathbf{pref}(\xi) \subseteq \mathbf{pref}(F)\}$

Fact

$F \subseteq X^\omega$ is closed if and only if $\mathbf{pref}(\xi) \subseteq \mathbf{pref}(F)$ implies $\xi \in F$.

$\{0,1\}^\omega$ as a Tree



The BOREL-Hierarchy: First Levels

Open Sets: $W \cdot X^\omega$

Closed Sets: $F = \mathcal{C}(F)$, $F = X^\omega \setminus W \cdot X^\omega$ [$\mathbb{F}$ – ferme, *fr.*]

$\mathbb{F}_\sigma$-Sets: $\bigcup_{i \in \mathbb{N}} F_i$ (F_i closed) [$\sigma \approx \sum$ – sum]

$\mathbb{G}_\delta$-Sets: $\bigcap_{i \in \mathbb{N}} E_i$ (E_i open), $\bigcap_{i \in \mathbb{N}} W_i \cdot X^\omega$

[δ – Durchschnitt, *german* for intersection]

$\Longrightarrow$

	Example	Closure properties	
Open sets	$0^*1 \cdot X^\omega$	$\cap$	$\bigcup$
Closed sets	$\{0^\omega\}$	$\cup$	$\bigcap$
$\mathbb{F}_\sigma$-sets	$\{0,1\}^* \cdot 0^\omega$	$\cap$	$\bigcup_{i \in \mathbb{N}}$
$\mathbb{G}_\delta$-sets	$(0^*1)^\omega$	$\cup$	$\bigcap_{i \in \mathbb{N}}$

The BOREL-Hierarchy of ω-languages

$$
\begin{array}{ccccc}
 & & \vdots & & \\
 & \nwarrow & & \nearrow & \\
 & & \mathbb{F}_{\sigma\delta} \cap \mathbb{G}_{\delta\sigma} & & \\
 & & \uparrow & & \\
 & & \mathrm{Bool}(\mathbb{G}_\delta) & & \\
 & \nearrow & & \nwarrow & \\
\{F : X^\omega \setminus F \in \mathbb{G}_\delta\} = \mathbb{F}_\sigma & & & & \mathbb{G}_\delta \\
 & \nwarrow & & \nearrow & \\
 & & \mathbb{F}_\sigma \cap \mathbb{G}_\delta & & \\
 & & \uparrow & & \\
 & & \mathrm{Bool}(\mathbb{G}) & & \\
 & \nearrow & & \nwarrow & \\
\{F : X^\omega \setminus F \in \mathbb{G}\} = \mathbb{F} & & & & \mathbb{G} \quad \text{open sets} \\
 & \nwarrow & & \nearrow & \\
 & & \mathbb{F} \cap \mathbb{G} & &
\end{array}
$$

Automata on ω-words: BÜCHI-automata

$\mathcal{A} = (Q, \Delta, q_0, Q_{\mathrm{fin}})$ is a BÜCHI-*Automaton* over X : $\Longleftrightarrow$

1. Q is a non-empty set (*states*)
2. $q_0 \in Q$ (*initial state*)
3. $\Delta \subseteq Q \times X \times Q$ (*transitions*)
4. $Q_{\mathrm{fin}} \subseteq Q$ (*final states*)

$\Longrightarrow$

- $\mathcal{A}$ is a *finite* automaton, if Q is finite.
- $\mathcal{A}$ is a *deterministic* automaton, if $(q, x, q'), (q, x, q'') \in \Delta$ implies $q' = q''$.

Büchi-automata: Acceptance

Run on ξ: $(q_i)_{i\in\mathbb{N}}$ with $\forall i \geq 0 : (q_i, \xi(i+1), q_{i+1}) \in \Delta$

$$\begin{array}{ccccccccccc} q_0 & & q_1 & & \color{red}{q_2} & & \color{red}{q_{i-1}} & & q_i & & \\ & \searrow & \uparrow & \searrow & \uparrow & \cdots & \uparrow & \searrow & \uparrow & \searrow & \cdots \\ & & \xi(1) & & \xi(2) & & \xi(i-1) & & \xi(i) & & \end{array}$$

$\Longrightarrow$

$\mathcal{A}$ accepts ξ: $\exists (q_i)_{i\in\mathbb{N}} \quad \forall i \geq 0 : (q_i, \xi(i+1), q_{i+1}) \in \Delta \quad \wedge$

$\exists^\infty k : q_k \in Q_{\mathrm{fin}}$

$\mathcal{A}$ accepts F: $F = \{\xi : \mathcal{A} \text{ accepts } \xi\}$

Automata on ω-words

Other types of ω-automata

- MULLER-automata
- RABIN-automata
- STREETT-automata

- The difference consists in acceptance conditions.
- Deterministic variants are as powerful as non-deterministic BÜCHI-automata.

Automata on ω-words: MULLER-automata

$\mathcal{A} = (Q, \Delta, q_0, \mathcal{T})$ is a MULLER-*Automaton* over X : $\Longleftrightarrow$

1. Q is a non-empty set (*states*)
2. $q_0 \in Q$ (*initial state*)
3. $\Delta \subseteq Q \times X \times Q$ (*transitions*)
4. $\mathcal{T} \subseteq 2^Q$ (*table of final sets*)

$\Longrightarrow$

$\mathcal{A}$ accepts ξ: $\exists (q_i)_{i\in\mathbb{N}} \quad \forall i \geq 0 : (q_i, \xi(i+1), q_{i+1}) \in \Delta \quad \wedge$
$\{q : \exists^\infty k (q_k = q)\} \in \mathcal{T}$

$\mathcal{A}$ accepts F: $F = \{\xi : \mathcal{A} \text{ accepts } \xi\}$

Regular ω-languages

Definition (Regular ω-language)

An ω-language $F \subseteq X^{\omega}$ is called *regular* if and only if F is accepted by a finite automaton

Theorem (Büchi 1962)

1. *An ω-language $F \subseteq X^{\omega}$ is regular if and only if*

$$F = \bigcup_{i=1}^{n} W_i \cdot V_i^{\omega}$$

   for some $n \in \mathbb{N}$ and regular languages $W_i, V_i \subseteq X^*$.

2. *The set of regular ω-languages over X is closed under Boolean operations.*

Ultimately Periodic ω-words

Definition (Ultimately periodic ω-words)

$\mathrm{Ult} := \{w \cdot v^{\omega} : w, v \in X^* \wedge v \neq e\}$ the set of *ultimately periodic ω-words*.

Theorem (Büchi 1962)

1. *Every non-empty regular ω-language contains an ultimately periodic ω-word.*
2. *Let $E, F \subseteq X^{\omega}$ be regular. Then*

$$E = F \iff E \cap \mathrm{Ult} = F \cap \mathrm{Ult}.$$

Lemma

If $F \subseteq X^{\omega}$ is regular then its prefix language $\mathbf{pref}(F) \subseteq X^*$ *and its closure* $\mathcal{C}(F)$ *are also regular, and if $W \subseteq X^*$ is a regular language, then $W \cdot F$ is regular.*

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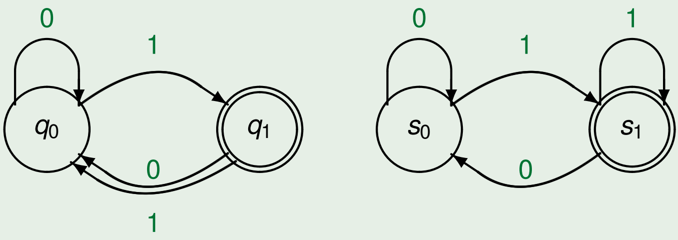
Automata and Topology: Deterministic ω-languages

Theorem (Landweber 1969)

An ω-language $F \subseteq X^\omega$ is accepted by a finite deterministic BÜCHI*-automaton (DBA) if and only if F is regular and a $\mathbb{G}_\delta$-set.*
[F is *deterministic* regular.]

$\Longrightarrow$

Example (Two automata accepting $F = (0^*1)^\omega$ (Muller 1963))



Automata and Topology: Closed ω-languages

Definition (Trim Automaton)

A *trim automaton* is a partial automaton $\mathcal{A} = (X, Q, \Delta, q_0, Q_{\text{fin}})$ with $Q_{\text{fin}} = Q$.

Lemma

An ω-language $F \subseteq X^\omega$ is accepted by a finite (deterministic) trim automaton (TA) if and only if F is regular and closed in CANTOR *space.*

Lemma

An ω-language $F \subseteq X^\omega$ is regular and closed in CANTOR *space if and only if* $\mathbf{pref}(F)$ *is a regular language and* $F = \{\xi : \mathbf{pref}(\xi) \subseteq \mathbf{pref}(F)\}$.

Weak BÜCHI Automata: Co-deterministic ω-languages

Definition (Weak BÜCHI automata)

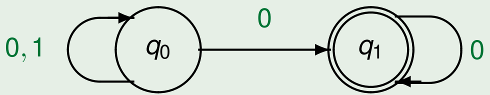
An automaton $\mathcal{A} = (X, Q, \Delta, q_0, Q_{\text{fin}})$ is referred to as a *weak* BÜCHI *automaton* provided Q_{fin} is a union of strongly connected components.

Theorem (*St.* and Wagner 1974, Wagner 1979)

$F \subseteq X^\omega$ is accepted by a finite weak BÜCHI *automaton (NWA) if and only if F is regular and an $\mathbb{F}_\sigma$-set.*

$\Longrightarrow$

Example (An automaton accepting $\{0,1\}^* \cdot 0^\omega$)



Weak BÜCHI Automata and BOREL hierarchy

Theorem (*St.* and Wagner 1974, Wagner 1979)

1. *$F \subseteq X^{\omega}$ is accepted by a finite deterministic weak* BÜCHI *automaton (DWA) if and only if F is regular and simultaneously an $\mathbb{F}_{\sigma}$- and a $\mathbb{G}_{\delta}$-set.*
2. *If $F \subseteq X^{\omega}$ is regular and simultaneously an $\mathbb{F}_{\sigma}$- and a $\mathbb{G}_{\delta}$-set then it is a Boolean combination of open regular ω-languages.*

$\Longrightarrow$

Theorem (*St.* and Wagner 1974, Wagner 1979)

Given a deterministic BÜCHI*-automaton, it is decidable in polynomial time whether the accepted ω-language is simultaneously an $\mathbb{F}_{\sigma}$- and a $\mathbb{G}_{\delta}$-set.*

The BOREL-Hierarchy of Regular ω-languages

$$
\begin{array}{ccccc}
 & & \mathrm{Bool}(\mathbb{G}_\delta^R) & & (=\ \mathrm{NBA}) \\
 & \nearrow & & \nwarrow & \\
(\mathrm{NWA}\ =)\ \mathbb{F}_\sigma^R & & & & \mathbb{G}_\delta^R\ (=\ \mathrm{DBA}) \\
 & \nwarrow & & \nearrow & \\
 & & \mathrm{Bool}(\mathbb{G}^R) = \mathbb{F}_\sigma^R\ \cap\ \mathbb{G}_\delta^R & & (=\ \mathrm{DWA}) \\
 & \nearrow & & \nwarrow & \\
(\mathrm{TA}\ =)\ \mathbb{F}^R & & & & \mathbb{G}^R \\
 & \nwarrow & & \nearrow & \\
 & & \mathbb{F} \cap \mathbb{G} & &
\end{array}
$$

MULLER-automata: Topology and Tables

Theorem (Landweber 1969, *St.* and Wagner 1974, Wagner 1979)

Let $\mathcal{A} = (Q, \Delta, q_0, \mathcal{T})$ be a MULLER*-automaton and F be accepted by $\mathcal{A}$.*

1. *If $\mathcal{A}$ is deterministic and $\mathcal{T}$ is upwardly closed ($Q' \in \mathcal{T} \wedge Q' \subseteq Q'' \to Q'' \in \mathcal{T}$) then $F \in \mathbb{G}_\delta$.*
2. *If $\mathcal{T}$ is downwardly closed ($Q' \in \mathcal{T} \wedge Q' \supseteq Q'' \to Q'' \in \mathcal{T}$) then $F \in \mathbb{F}_\sigma$.*

Theorem (Landweber 1969, *St.* and Wagner 1974, Wagner 1979)

Let F be a regular ω-language.

1. *If $F \in \mathbb{G}_\delta$ then there is a deterministic* MULLER*-automaton $\mathcal{A} = (Q, \Delta, q_0, \mathcal{T})$ with upwardly closed $\mathcal{T}$ accepting F.*
2. *If $F \in \mathbb{F}_\sigma$ then there is a deterministic* MULLER*-automaton $\mathcal{A} = (Q, \Delta, q_0, \mathcal{T})$ with downwardly closed $\mathcal{T}$ accepting F.*

Characterisation by Regular Languages

	CLASS	REPRESENTATION	COMMENT
1.	$\mathbb{G}^R$	$W \cdot X^\omega$	W regular (and prefix-free)
2.	$\mathbb{F}^R$	$\{\xi : \mathbf{pref}(\xi) \subseteq W\}$	W regular
3.	$\mathbb{F}^R_\sigma \cap \mathbb{G}^R_\delta$	$\bigcup_{i=1}^{n} W_i \cdot F_i$	$F_i \subseteq X^\omega$ closed, W_i, F_i regular and W_i prefix-free
4.	$\mathbb{F}^R_\sigma$	$\bigcup_{i=1}^{n} W_i \cdot F_i$	$F_i \subseteq X^\omega$ closed, W_i, F_i regular
5.	$\mathbb{G}^R_\delta$	$\bigcup_{i=1}^{n} W_i \cdot V_i^\omega$	W_i, V_i regular and prefix-free
6.	regular	$\bigcup_{i=1}^{n} W_i \cdot V_i^\omega$	W_i, V_i regular (and V_i prefix-free)

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Nerode Right Congruence

Definition

$$u \sim_W v \quad :\Leftrightarrow \quad \forall w(w \in X^* \rightarrow (u \cdot w \in W \longleftrightarrow v \cdot w \in W))$$
$$[v]_{\sim_W} \quad := \quad \{u : u \sim_W v\} \quad \text{[equivalence classes]}$$
$$\mathrm{Ind}(\sim_W) \quad := \quad |\{[v]_{\sim_W} : v \in X^*\}|$$

Theorem (folklore)

$W \subseteq X^*$ is regular if and only if $\mathrm{Ind}(\sim_W) < \infty$.

Left Derivative of Languages and ω-languages

Definition (Left derivative)

Let $B \subseteq X^* \cup X^\omega$ and $w \in X^*$.

$$B/w := \{\eta : w \cdot \eta \in B\}.$$

Property

$$B/v = B/w \iff v \sim_B w \text{ and } \left|\{B/w : w \in X^*\}\right| = \mathrm{Ind}(\sim_B)$$

$\Longrightarrow$

Definition (Associated automaton)

$\mathcal{A}_B = (\{B/w : w \in X^*\}, \Delta_B, B/e)$ where

$$\Delta_B = \{(B/w, x, B/wx) : w \in X^* \wedge x \in X\}$$

Theorem (folklore)

If $W \subseteq X^*$ then $\mathcal{A}_W = (\{W/v : v \in X^*\}, \Delta_B, B/e, \{W/u : u \in W\})$ is a minimal deterministic automaton accepting W.

NERODE Right Congruence for ω-languages

Definition: $u \sim_F v :\Longleftrightarrow \forall \xi (\xi \in X^\omega \rightarrow (u\cdot\xi \in F \longleftrightarrow v\cdot\xi \in F))$

Theorem (Trakhtenbrot 1962, Jürgensen and Thierrin 1983)

1. [Tr] *If $F \subseteq X^\omega$ is regular, then $\sim_F$ has finite index* $(\mathrm{Ind}(\sim_F) < \infty)$.
2. [Tr] *If $\mathrm{Ind}(\sim_F) < \infty$ and $F \subseteq X^\omega$ is closed then F is regular.*
3. [JT] *There are $2^{2^{\aleph_0}}$ ω-languages E with $\mathrm{Ind}(\sim_E) = 1$.*

$\Longrightarrow$

Theorem (1983)

Let $F \subseteq X^\omega$ be in $\mathbb{F}_\sigma \cap \mathbb{G}_\delta$ and $\mathrm{Ind}(\sim_F) < \infty$. Then

1. *F is already regular and*
2. *F is accepted by its associated automaton $\mathcal{A}_F$.*

Minimisation of ω-automata

Lemma

If $F \subseteq X^\omega$ then every deterministic Büchi- (Muller-)*automaton accepting F has $\mathcal{A}_F$ as a homomorphic image.*

Corollary

*If a deterministic (co-)*Büchi- (Muller-)*automaton $\mathcal{A}$ accepts $F \subseteq X^\omega$ then $\mathcal{A}$ has at least $\mathrm{Ind}(\sim_F)$ states.*

$\Longrightarrow$

Fact

1. *There are regular ω-languages $F \subseteq X^\omega$ having more than one minimal-state* Büchi- (Muller-)*automaton $\mathcal{A}$ accepting F.*
2. *There are regular ω-languages $F \subseteq X^\omega$ having exactly one minimal-state* Büchi- (Muller-)*automaton $\mathcal{A}$ accepting F but not being accepted by $\mathcal{A}_F$.*

Minimisation of ω-automata: $n \log n$-algorithm

Theorem (Löding 2001)

There is an algorithm minimising an n-state deterministic weak BÜCHI *automaton accepting an ω-language F in $O(n \log n)$ time to the associated automaton $\mathcal{A}_F$.*

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  English translation: AMS Transl. 59 (1966), 23–55.)

Small and Large Sets in CANTOR Topology

dense: $\mathcal{C}(F) = X^\omega$, $\mathbf{pref}(F) = X^*$

nowhere dense: $w \cdot X^\omega \not\subseteq \mathcal{C}(F)$ for all $w \in X^*$

The closure does not contain an open set.

First BAIRE category *or* meagre: $\bigcup_{i \in \mathbb{N}} F_i$ (F_i nowhere dense)

Second BAIRE category: `not` of first BAIRE category

residual: $X^\omega \setminus F$ is of first BAIRE category

Small and Large Sets

	Topology	Closure properties	
very large	F is residual	superset	$\bigcap_{i\in\mathbb{N}}$
large	F is of 2^{nd} BAIRE category	superset	–
small	F is of 1^{st} BAIRE category	subset	$\bigcup_{i\in\mathbb{N}}$
very small	F is nowhere dense	subset	$\bigcup$

$\Longrightarrow$

	Logical description	Example
very large	infinitely many ones	$(0^*1)^\omega$
large		$0(0^*1)^\omega \cup 1\{0,1\}^* \cdot 0^\omega$
small	finitely many ones	$\{0,1\}^* \cdot 0^\omega$
very small	$\leq n$ ones	$\bigcup_{i=0}^{n}(0^*1)^i \cdot 0^\omega$

Small and Large Sets: BOREL classes

Lemma

In every complete metric space $(\mathcal{X}, \rho)$ the following are true.

1. *Every nowhere dense set is contained in a closed nowhere dense set.*
2. *Every set of* 1^{st} BAIRE *category is a subset of an* $\mathbb{F}_\sigma$*-set of* 1^{st} BAIRE *category.*
3. *Every* $\mathbb{G}_\delta$*-set of* 1^{st} BAIRE *category is nowhere dense.*
4. *If M is a* $\mathbb{G}_\delta$*-set then* $\mathcal{C}_\rho(M) \setminus M$ *is a set of* 1^{st} BAIRE *category.*
5. *Every residual set contains a residual* $\mathbb{G}_\delta$*-set.*
6. *Every residual set is dense.*

Small Regular ω-languages

Example (Forbidden subwords)

$E = X^\omega \setminus X^* \cdot v \cdot X^\omega$ is nowhere dense because $E \cap w \cdot v \cdot X^\omega = \emptyset$ for all $w \in X^*$.

$\Longrightarrow$

Theorem (1976)

Let $F \subseteq X^\omega$ be a regular ω-language.

1. *F is nowhere dense if and only if there is a $v \in X^*$ such that $F \subseteq X^\omega \setminus X^* \cdot v \cdot X^\omega$.*
2. *F is of 1^{st} BAIRE category if and only if $F \subseteq \bigcup_{v \in X^*} (X^\omega \setminus X^* \cdot v \cdot X^\omega)$.*

Visualisation: r-adic Expansion

$$Y = \{0, 1, \ldots, r-1\}$$

$$0.\eta \in [0,1] \subseteq \mathbb{R} \quad \xleftarrow{\nu_r} \quad \eta \in Y^\omega$$

$$(0.\operatorname{proj}_1 \xi, \ldots, 0.\operatorname{proj}_d \xi) \in [0,1]^d \quad \xleftarrow{\nu_r} \quad \xi \in (\underbrace{Y \times \ldots \times Y}_{d-\text{times}})^\omega$$

Example: $r = 2$

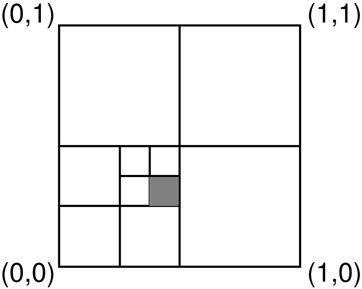
$$\frac{3}{4} \quad \xleftarrow{\nu_2} \quad \begin{cases} 0.11000\ldots \\ 0.10111\ldots \end{cases}$$

$$(x_\beta, y_\beta) \in [0,1]^2 \quad \xleftarrow{\nu_2} \quad \beta \in \{(0,0), \ldots, (1,1)\}^\omega$$

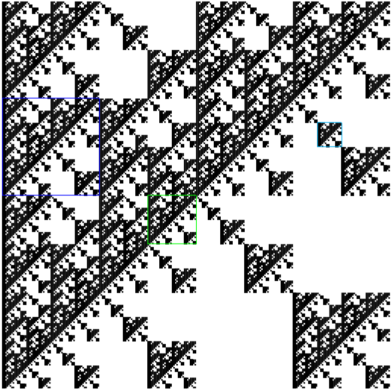
$$\begin{matrix} x_\beta = 0.x_1 x_2 x_3 \ldots \\ y_\beta = 0.y_1 y_2 y_3 \ldots \end{matrix} \quad \xleftarrow{\nu_2} \quad \beta = \begin{pmatrix} x_1 \\ y_1 \end{pmatrix} \begin{pmatrix} x_2 \\ y_2 \end{pmatrix} \begin{pmatrix} x_3 \\ y_3 \end{pmatrix} \ldots$$

Visualisation in the Unit Square $[0,1]^2$

Notation ($r = 2$): $X := \{(0,0),(0,1),(1,0),(1,1)\}$
Quadrant in $[0,1]^2$: $\mathbf{Q}_{(0,0)(1,1)(1,0)} = \nu_2((0,0)(1,1)(1,0)\cdot X^\omega)$



Visualisation: A Regular Nowhere Dense Set



$u = (1,1)\cdot(0,1)\cdot(1,0)\cdot(0,1)$

$w = (1,0)\cdot(0,0)$

$v = (0,0)\cdot(1,1)\cdot(1,1)$

$$
\begin{aligned}
S_1 &= (0,1)\cdot S_3 \cup (0,0)\cdot S_1 \cup (1,1)\cdot S_1 \cup (1,0)\cdot S_2\\
S_2 &= (0,1)\cdot S_2 \cup (0,0)\cdot S_1 \cup (1,1)\cdot S_3 \cup (1,0)\cdot S_1\\
S_3 &= (0,1)\cdot S_1 \cup (1,0)\cdot S_3
\end{aligned}
$$

BERNOULLI Measures $\bar{\mu}$ on X^ω

BERNOULLI measure on X^*: $\mu : X^* \to [0,1]$

- $\sum_{x\in X} \mu(x) = 1$, $\mu(x) > 0$;
- $\mu(w \cdot v) := \mu(w) \cdot \mu(v)$

$\Longrightarrow$

Property

If $W \subseteq X^*$ is prefix-free then $\sum_{w\in W} \mu(w) \leq 1$

$\Longrightarrow$

Definition (BERNOULLI measure on X^ω)

Measure on balls: $\bar{\mu}(w \cdot X^\omega) := \mu(w)$

Measure on open sets: If $W \subseteq X^*$ is prefix-free then

$$\bar{\mu}(W \cdot X^\omega) := \sum_{w\in W} \mu(w)$$

Comparison of Small and Large Sets

	Measure $[\mu(X^\omega) = 1]$	**Topology**
very large	$\bar{\mu}(F) = 1$	F is residual
large	$\bar{\mu}(F) > 0$ *or* F is not measurable	F is of 2nd BAIRE category
small	$\bar{\mu}(F) = 0$	F is of 1st BAIRE category
very small	$\bar{\mu}(\mathcal{C}(F)) = 0$ $\Longrightarrow$	F is nowhere dense

$\Longrightarrow$

Proposition (Incomparability (cf. OXTOBY: *Measure and Category*))

1. *There is a nowhere dense set $F \subseteq X^\omega$ such that $\bar{\mu}(F) > 0$.*
2. *There is a set of 1st* BAIRE *category such that $\bar{\mu}(F) = 1$.*

   There is a residual set $E \subseteq X^\omega$ such that $\bar{\mu}(E) = 0$.

Probabilistic Arguments

Bad news for probabilistic arguments

Example

The set of BERNOULLI- (BOREL-normal) sequences over X is of 1^{st} BAIRE category.

An Example: Łukasiewicz Language

Defining equation:

$$\begin{aligned} Ł &= 0 \cup 1 \cdot Ł^3 \\ \mu(Ł) &= \mu(0) + \mu(1) \cdot \mu(Ł^3) \end{aligned}$$

$\Longrightarrow$

1. Ł is a simple deterministic context-free language, hence prefix-free.

→ $\mu(Ł^n) = \mu(Ł)^n$

2. Eq. (2) $\mu(Ł) = (1 - \mu(1)) + \mu(1) \cdot \mu(Ł^3)$ has the positive solutions $t_0 = 1$ and $t_1 = -\frac{1}{2} + \sqrt{\frac{1}{\mu(1)} - \frac{3}{4}}$.
3. $\mu(Ł)$ is the smallest positive solution of Eq. (2).
4. $\mu(Ł) = \frac{\sqrt{5}-1}{2} < 1$ for $\mu(1) = \frac{1}{2}$ and $\mu(Ł) = 1$ for $\mu(1) \leq \frac{1}{3}$.

→ $\bar{\mu}(\bigcap_{n\in\mathbb{N}} Ł^n \cdot X^\omega) = \begin{cases} 0 & \text{for } \mu(1) = 1/2 \text{ , and} \\ 1 & \text{for } \mu(1) \leq 1/3 \end{cases}$

5. $\bigcap_{n\in\mathbb{N}} Ł^n \cdot X^\omega$ is a $\mathbb{G}_\delta$-set and dense in $\{0,1\}^\omega$, thus its complement is of 1st BAIRE category.

Topology and Measure in CANTOR space

Theorem (1976)

Let $\mu : X^* \to (0,1)$ be a BERNOULLI measure and let $F \subseteq X^\omega$ be a regular ω-language.
Then F is of 1^{st} BAIRE category if and only if $\bar{\mu}(F) = 0$.

$\Longrightarrow$

Proof Scheme: Induction on BOREL CLASSES

closed	F is nowhere dense if and only if $\bar{\mu}(F) = 0$.
$\mathbb{F}_\sigma$-sets	F is a countable union of closed regular ω-languages.
$\mathbb{G}_\delta$-sets	$\mathcal{C}(F)$ is the union of F and $\mathcal{C}(F) \setminus F$, where $\mathcal{C}(F) \setminus F$ is a regular ω-language in $\mathbb{F}_\sigma$ of 1^{st} BAIRE category.
general	F is a countable union of regular ω-languages in $\mathbb{G}_\delta$.

Balanced Measures

Definition

A finite measure $\bar{\mu}$ on X^{ω} is called *balanced* if the following holds true.

$$\exists c > 0\ \forall w \in X^* \forall x \in X: \begin{array}{rcl} \bar{\mu}(wx \cdot X^{\omega}) & > & c \cdot \bar{\mu}(w \cdot X^{\omega}) \quad \text{or} \\ \bar{\mu}(wx \cdot X^{\omega}) & = & 0 \end{array}$$

Definition (Support)

Let $\bar{\mu}$ be a finite measure on X^{ω}.
The smallest closed set F with $\bar{\mu}(F) = \bar{\mu}(X^{\omega})$ is referred to as the *support* $\mathbf{supp}(\bar{\mu})$ of $\bar{\mu}$.

Relativisation: Nowhere dense sets

Definition (Relative density)

Let $S, F \subseteq X^{\omega}$, $S \neq \emptyset$. We call F nowhere dense in S if for every non-empty ball $S \cap w \cdot X^{\omega}$ in S there is a non-empty sub-ball $S \cap w \cdot v \cdot X^{\omega}$ disjoint with F.

$\Longrightarrow$

Lemma (1998)

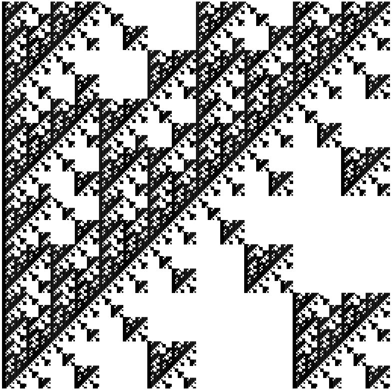
Let $S \subseteq X^{\omega}$ be a regular ω-language.
A regular ω-language $F \subseteq X^{\omega}$ is nowhere dense in S if and only if for every $w \in \mathbf{pref}(S)$ there is a $v \in X^{*}$ such that

1. $|v| < \mathrm{Ind}(\sim_F) \cdot \mathrm{Ind}(\sim_S) + 1$ *and*
2. *and* $w \cdot v \in \mathbf{pref}(S)$ *and* $w \cdot v \notin \mathbf{pref}(F)$.

Remark

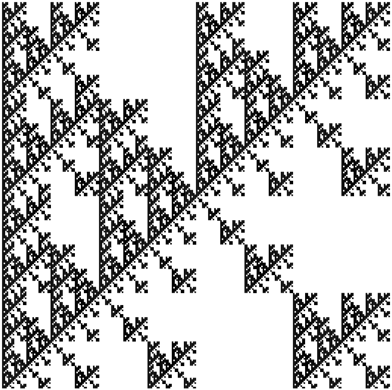
Observe that $S \cap w \cdot X^{\omega} \neq \emptyset$ if and only if $w \in \mathbf{pref}(S)$.

Visualisation: ω-language S



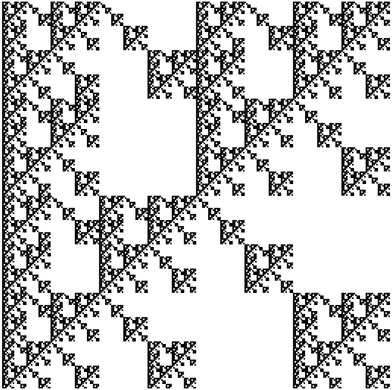
$$\begin{array}{llllll} S_1 & = & (0,1)\cdot S_3 & \cup\,(0,0)\cdot S_1 & \cup\,(1,1)\cdot S_1 & \cup\,(1,0)\cdot S_2 \\ S_2 & = & (0,1)\cdot S_2 & \cup\,(0,0)\cdot S_1 & \cup\,(1,1)\cdot S_3 & \cup\,(1,0)\cdot S_1 \\ S_3 & = & (0,1)\cdot S_1 & & & \cup\,(1,0)\cdot S_3 \end{array}$$

Visualisation: F_0 nowhere dense in S



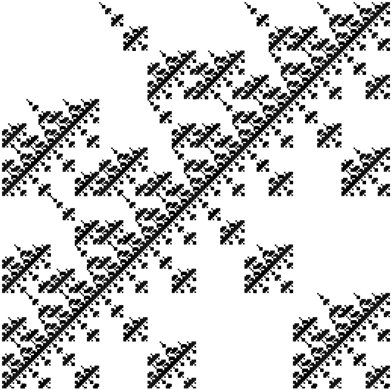
$$\begin{aligned}
S_1 &= (0,1)\cdot S_3 \cup (0,0)\cdot S_1 \cup (1,1)\cdot S_1 \cup (1,0)\cdot S_2\\
S_2 &= (0,1)\cdot S_2 \cup (0,0)\cdot S_1 \cup (1,1)\cdot \emptyset \cup (1,0)\cdot S_1\\
S_3 &= (0,1)\cdot S_1 \cup (1,0)\cdot S_3
\end{aligned}$$

Visualisation: F_1 nowhere dense in S



$$
\begin{aligned}
S_1 &= (0,1)\cdot S_3 \;\cup (0,0)\cdot S_1 \;\cup (1,1)\cdot S_1 \;\cup (1,0)\cdot S_2\\
S_2 &= (0,1)\cdot \emptyset \;\cup (0,0)\cdot S_1 \;\cup (1,1)\cdot S_3 \;\cup (1,0)\cdot S_1\\
S_3 &= (0,1)\cdot S_1 \;\cup (1,0)\cdot S_3
\end{aligned}
$$

Visualisation: F_2 nowhere dense in S



$$\begin{aligned}
S_1 &= (0,1)\cdot S_3 \cup (0,0)\cdot S_1 \cup (1,1)\cdot S_1 \cup (1,0)\cdot S_2\\
S_2 &= (0,1)\cdot S_2 \cup (0,0)\cdot S_1 \cup (1,1)\cdot S_3 \cup (1,0)\cdot \emptyset\\
S_3 &= (0,1)\cdot S_1 \cup (1,0)\cdot S_3
\end{aligned}$$

Relativisation: Inhomogeneity

Example (Inhomogeneity)

For $S = 0\cdot(0\cdot X)^\omega \cup 1\cdot X^\omega$ we have:
$F_1 = 0\cdot(0\cdot X)^\omega$ is of 2nd BAIRE category in S, and
$F_2 = 1\cdot(0\cdot X)^\omega$ is nowhere dense in S.

Relativisation: Topology and Measure in CANTOR space

Theorem (*St.* 1998, Varacca and Völzer 2006)

Let $S \subseteq X^\omega$ regular and closed and $F \subseteq S$ be regular. Then the following are equivalent.

1. *F ist is of 1^{st} BAIRE category in S.*
2. *There is a balanced finite measure $\bar{\mu}$ with support $\mathbf{supp}(\bar{\mu}) = S$ such that $\bar{\mu}(F) = 0$.*
3. *$\bar{\mu}(F) = 0$ for all balanced finite measures $\bar{\mu}$ with support $\mathbf{supp}(\bar{\mu}) = S$.*

$\Longrightarrow$

Corollary

Let $S \subseteq X^\omega$ regular and closed. Then

$$\bigcup\{F : F \subseteq S \wedge F \text{ is regular and nowhere dense in } S\}$$

is a null-set universal for all balanced finite measures $\bar{\mu}$ with support $\mathbf{supp}(\bar{\mu}) = S$.

References: Automata and Measure

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What ω-automata cannot prove

ω-automata cannot prove that

1. there are sets in BOREL classes higher than $\mathrm{Bool}(\mathbb{G}_\delta)$,
2. there are sets in $(\mathbb{G}_\delta \cap \mathbb{F}_\sigma) \setminus \mathrm{Bool}(\mathbb{G})$,
3. there are nowhere dense BERNOULLI non-nullsets,
4. there are BERNOULLI nullsets of 2nd BAIRE category,
5. there are sets which are BERNOULLI nullsets w.r.t. measure $\bar{\mu}_1$ but not w.r.t. measure $\bar{\mu}_2$

$\Longrightarrow$

but they are useful for proving largeness by probabilistic arguments.

Thank you

$\Longrightarrow$