Liouville, Computable, Borel Normal and Martin-Löf Random Numbers

Cris Calude
University of Auckland

Ludwig Staiger
Martin-Luther-Universität Halle-Wittenberg

Mathematics for Computation
Benediktinerabtei St. Mauritius, Niederalteich
8-13 May 2016, Lower Bavaria

Outline

Notation: Strings and Languages

Finite Alphabet $A_b = \{0, \ldots, b-1\}$, cardinality $|A_b| = b$

Finite strings (words) $w = x_1 \cdots x_n \in A_b^*$, $x_i \in A_b$

Length $|w| = n$

Languages $W \subseteq A_b^*$

Infinite strings (ω-words) $\mathbf{x} = x_1 \cdots x_n \cdots \in A_b^\omega$

Prefixes of infinite strings $\mathbf{x}\restriction n \in A_b^*$, $|\mathbf{x}\restriction n| = n$

ω-Languages $F \subseteq A_b^\omega$

Real Numbers and Expansions

Basis: $b \in \mathbb{N}$, $b \geq 2$

b-ary expansion: $\alpha = 0.\mathbf{x}$, $\mathbf{x} = x_1 x_2 \cdots x_i \cdots \in A_b^\omega$, $\alpha \in [0,1]$

$\alpha = \sum_{i=1}^{\infty} x_i \cdot b^{-i}$

Fact

Let $b \in \mathbb{N}$, $b \geq 2$ and $\alpha \in [0,1]$.

- *Every real number α has at least one and at most two b-ary expansions.*
- *If α has two b-ary expansions then $\alpha = m \cdot b^{-n}$ with $m \in \mathbb{Z}$ and $n \in \mathbb{N}$.*
- *Every irrational number has exactly one b-ary expansion.*

Computable Numbers

Definition (Computable Number)

A real number α is *computable* if and only if there is a computable function $f : \mathbb{N} \to \mathbb{Z} \times \mathbb{N}$ such that, for all $n \in \mathbb{N}$

$$|\alpha - \frac{p}{q}| \leq 2^{-n} \text{ whenever } f(n) = (p, q).$$

Fact

A real number α is computable if and only if,

- *for some b, it has a computable (as a function $\mathbf{x} : \mathbb{N} \setminus \{0\} \to A_b$) b-ary expansion, or*
- *for all b, its b-ary expansions are computable.*

Liouville Numbers

Definition (Liouville number)

A real number α is called a *Liouville number* if it is irrational and for every positive integer k, there exist integers p_k and q_k with $q_k > 1$ such that

$$\left|\alpha - \frac{p_k}{q_k}\right| < \frac{1}{q_k^k}.$$

Example

$\alpha = \sum_{n=0}^{\infty} 2^{-n!}$ is a Liouville number.

$$|\alpha - \sum_{n=0}^{k} 2^{-n!}| = |\alpha - \frac{\sum_{n=0}^{k} 2^{k!-n!}}{2^{k!}}| < \frac{1}{2^{(k+1)!-1}} \leq \frac{1}{(2^{k!})^k}$$

Irrationality Exponents

Definition (Irrationality exponent)

The *irrationality exponent* of a real number α is a measure of how "closely" α can be approximated by rationals:

$$\inf\left\{\mu \geq 0 : \left|\alpha - \frac{p}{q}\right| < \frac{1}{q^{\mu}} \text{ has finitely many solutions } p, q \in \mathbb{Z} \setminus \{0\}\right\}.$$

Thus, Liouville numbers are those reals having infinite irrationality exponent.

Borel Normal Numbers

Definition (Borel normality in base b)

A real number α is *Borel normal in base b* if and only if every word $w \in A_b^\ell$ of length ℓ appears with the same frequency as subword in the b-ary expansion $\mathbf{x}$ of α, that is, for $\mathbf{x} \in A_b^\omega$ with $\alpha = 0.\mathbf{x}$ it holds

$$\lim_{n\to\infty} \frac{\left|\{i : 1 \leq i \leq n \wedge \mathbf{x}\restriction i \in A_b^* \cdot w\}\right|}{n} = b^{-|w|}.$$

Definition (Absolute Borel normality)

A real number α is *Borel abolutely-normal* if and only if it is Borel normal in every base.

Liouville Numbers Borel Normal in Base b

Definition (De Bruijn Words)

A *de Bruijn word* $B \in A_b^*$ of order k is a shortest word having all words of length k in cyclic form as subwords.

$B(b,k)$ is the first word in lexicographic order having all words from A_b^k as subwords (in cyclic form).

Example (De Bruijn Words)

- $B(2,2) = 0011$ and $B(2,3) = 00010111$ are binary de Bruijn words of orders 2 and 3, respectively, and
- $B(3,2) = 001021122$ is a ternary de Bruijn word of order 2.

$$\color{red}{00}11, \quad 0\color{red}{01}1, \quad 00\color{red}{11}, \quad \color{red}{0}01\color{red}{1}$$

Fact

$$|B(b,k)| = b^k$$

Construction: Concatenating Finite Words

Let $f : \mathbb{N} \to \mathbb{N} \setminus \{0\}$ and $\mathcal{W} := (w_i)_{i \in \mathbb{N}},\ |w_i| > 0$.

$$\mathbf{x}_f(\mathcal{W}) := \underbrace{w_1 \cdots w_1}_{f(1) \text{ times}} \cdot \underbrace{w_2 \cdots w_2}_{f(2) \text{ times}} \cdot \ \cdots \ \cdot \underbrace{w_i \cdots w_i}_{f(i) \text{ times}} \cdot \ \cdots$$

Proposition (Maillet 1904, Nandakumar & Vangapelli 2014)

If $f(i) \geq i^i$ then

1. $0.\mathbf{x}_f(\mathcal{W})$ *is a Liouville number, and*
2. *if, moreover, $w_i = B(b, i)$ then $0.\mathbf{x}_f(\mathcal{W})$ is Borel normal in base b.*

Algorithmic Randomness

Unpredictability paradigm

An ω-word is *random* if and only if no constructive predicting (betting) strategy can win against it.

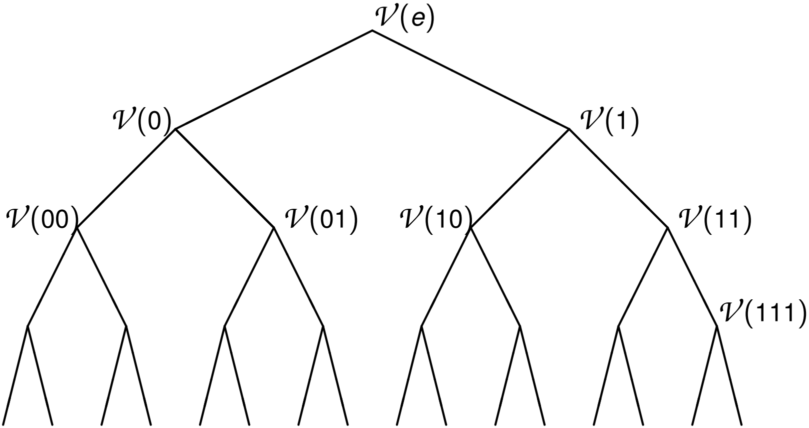
Our model:

- Playing against an ω-word $\mathbf{x} \in A_b^\omega$.
- Gambling strategy $\Gamma : A_b^* \times A_b \to [0,1]$ (bet on outcome $x \in A_b$) $\sum_{x \in A_b} \Gamma(w,x) \leq 1$ for $w \in A_b^*$
- $\mathcal{V}_\Gamma(\mathbf{x}{\restriction}n)$ is the capital after the nth round, that is,

$$\mathcal{V}_\Gamma(\mathbf{x}{\restriction}n+1) = b \cdot \Gamma(\mathbf{x}{\restriction}n, x) \cdot \mathcal{V}_\Gamma(\mathbf{x}{\restriction}n), \text{ for } \mathbf{x}(n+1) = x$$

- yields a (super-)martingale $\mathcal{V}_\Gamma : A_b^* \to \mathbb{R}_+$, that is,
$$\mathcal{V}_\Gamma(\mathbf{x}{\restriction}n) \geq \frac{1}{b} \cdot \sum_{x \in A_b} \mathcal{V}_\Gamma((\mathbf{x}{\restriction}n) \cdot x)$$

Gambling Strategies: Martingale $\mathcal{V} : \{0,1\}^* \to \mathbb{R}_+$



Randomness

Definition (Super-martingale property)

$$\mathcal{V}_\Gamma(w) \geq \frac{1}{b} \cdot \sum_{x \in A_b} \mathcal{V}_\Gamma(wx)$$

Theorem (Levin 1970)

For every A_b there is an optimal left-computable (computably approximable from below) super-martingale $\mathcal{U} : A_b^* \to \mathbb{R}_+$, that is, for all left-computable super-martingales $\mathcal{V} : A_b^* \to \mathbb{R}_+$ there is a constant $c > 0$ such that $\forall w \big(w \in A_b^* \to \mathcal{U}(w) \geq c \cdot \mathcal{V}(w)\big)$.

Definition (Martin-Löf Randomness)

$\mathbf{x} \in A_b^\omega$ is *Martin-Löf random* if and only if no constructive gambling strategy Γ can win against $\mathbf{x}$, that is, $\limsup_{n\to\infty} \mathcal{U}(\mathbf{x}{\restriction}n) < \infty$.

Partial Randomness: Definition

Partial randomness tries to measure, for $\mathbf{x} \in {A_b}^{\omega}$, the largest exponent γ with

$$\mathcal{U}(\mathbf{x}\restriction n) \approx b^{\gamma\cdot n+o(n)}.$$

More precisely, $\underline{\kappa}(\mathbf{x}) = 1-\gamma :\Longleftrightarrow$

$$\begin{aligned} \mathcal{U}(\mathbf{x}\restriction n) &\geq_{i.o.} b^{\gamma'\cdot n+o(n)} && \text{for } \gamma' < \gamma\text{, and} \\ \mathcal{U}(\mathbf{x}\restriction n) &\leq b^{\gamma'\cdot n+o(n)} && \text{for } \gamma' > \gamma. \end{aligned}$$

Fact

$$\underline{\kappa}(\mathbf{x}) = \liminf_{n\to\infty} \frac{n-\log_b \mathcal{U}(\mathbf{x}\restriction n)}{n} \in [0,1]$$

Observe

The higher the value $\underline{\kappa}(\mathbf{x})$ the '*more random*' the b-ary expansion $\mathbf{x}$.

Partial Randomness: Examples

Example (Dilution)

Let $\mathbf{x} = x_1 x_2 \cdots x_i \cdots \in A_b^\omega$ and $\mathbf{y} := x_1 00 x_2 00 \cdots x_i 00 \cdots$.

Then $\mathcal{U}(\mathbf{y}{\restriction}3n) \approx b^{2n} \cdot \mathcal{U}(\mathbf{x}{\restriction}n)$, and $\underline{\kappa}(\mathbf{y}) = \frac{1}{3} \cdot \underline{\kappa}(\mathbf{x})$.

Example (Non-random ω-words with $\underline{\kappa}(\mathbf{x}) = 1$)

Let $\mathbf{x} = x_1 x_2 \cdots x_i \cdots \in A_b^\omega$ be Martin-Löf random, and define

$$\mathbf{y}(i) := \begin{cases} 0 & \text{if } i \in \{2^n : n \in \mathbb{N}\}, \text{ and} \\ \mathbf{x}(i) & \text{otherwise.} \end{cases}$$

Then $\mathcal{U}(\mathbf{y}{\restriction}2^n) \geq c \cdot b^n$.

Partial Randomness: Base Independence

Theorem (Calude & Jürgensen 1995, Hertling & Weihrauch 2003, *St.* 2002)

Let $\mathbf{x} \in A_b^\omega$ *and* $\mathbf{y} \in A_r^\omega$ *and* $0.\mathbf{x} = 0.\mathbf{y}$.
Then $\mathbf{x}$ *is random if and only if* $\mathbf{y}$ *is random.*

Theorem (*St.* 2002)

Let $\mathbf{x} \in A_b^\omega$ *and* $\mathbf{y} \in A_r^\omega$ *and* $0.\mathbf{x} = 0.\mathbf{y}$.
Then $\underline{\kappa}(\mathbf{x}) = \underline{\kappa}(\mathbf{y})$.

Fact

1. *If* $\mathbf{x} \in A_b^\omega$ *is computable then* $\underline{\kappa}(\mathbf{x}) = 0$.
2. *If* $\mathbf{x} \in A_b^\omega$ *is random then* $\underline{\kappa}(\mathbf{x}) = 1$.

Partial Randomness: Further Relations

Theorem (Kolmogorov 1968, *St.* 2002)

1. *If* $\underline{\kappa}(\mathbf{x}) = 1$ *then the real* $\alpha = 0.\mathbf{x}$ *is Borel absolutely-normal.*
2. *If* α *is a Liouville number and* $\mathbf{x}$ *its b-ary expansion then* $\underline{\kappa}(\mathbf{x}) = 0$.

Theorem (Calude & *St.* 2015, Jarník 1929, Ryabko 1986)

Let $\alpha \in [0,1]$ *be irrational with irrationality exponent* μ *and let* $\mathbf{x}$ *be its b-ary expansion. Then* $\underline{\kappa}(\mathbf{x}) \leq 2/\mu$.

Moreover, for every $\mu, \mu \in [2, \infty]$, *there is an* $\alpha \in [0,1]$ *with irrationality exponent* μ *such that its b-ary expansion* $\mathbf{x}$ *satisfies* $\underline{\kappa}(\mathbf{x}) = 2/\mu$

Relations between the Classes of Reals

Fact

- *Martin-Löf random reals are Borel absolutely-normal ($\mathcal{M} \subseteq \mathcal{N}$).*
- *Martin-Löf random reals and computable numbers are disjoint ($\mathcal{M} \cap \mathcal{C} = \emptyset$).*
- *Martin-Löf random reals and Liouville numbers are disjoint ($\mathcal{M} \cap \mathcal{L} = \emptyset$).*
- *The following seven out of the 16 Boolean combinations are empty:* $\bar{\mathcal{L}} \cap \bar{\mathcal{C}} \cap \bar{\mathcal{N}} \cap \mathcal{M}$, $\bar{\mathcal{L}} \cap \mathcal{C} \cap \bar{\mathcal{N}} \cap \mathcal{M}$, $\mathcal{L} \cap \bar{\mathcal{C}} \cap \bar{\mathcal{N}} \cap \mathcal{M}$, $\mathcal{L} \cap \mathcal{C} \cap \bar{\mathcal{N}} \cap \mathcal{M}$, $\bar{\mathcal{L}} \cap \mathcal{C} \cap \mathcal{N} \cap \mathcal{M}$, $\mathcal{L} \cap \mathcal{C} \cap \mathcal{N} \cap \mathcal{M}$, *and* $\mathcal{L} \cap \bar{\mathcal{C}} \cap \mathcal{N} \cap \mathcal{M}$.

Question

What about the remaining nine intersections?

Borel Normality of Liouville Numbers

Theorem (Bugeaud 2002)

There are uncountably many Borel absolutely-normal Liouville numbers, and there are uncountably many Liouville numbers not *normal in any base.*

Theorem (Becher, Heiber & Slaman 2014)

There are computable Borel absolutely-normal Liouville numbers.

Theorem (Martin 2001)

There is a computable Liouville number not *normal in any base.*

- The combinations $\mathcal{L} \cap \bar{\mathcal{C}} \cap \mathcal{N} \cap \bar{\mathcal{M}}$, $\mathcal{L} \cap \bar{\mathcal{C}} \cap \bar{\mathcal{N}} \cap \bar{\mathcal{M}}$, $\mathcal{L} \cap \mathcal{C} \cap \mathcal{N} \cap \bar{\mathcal{M}}$ and $\mathcal{L} \cap \mathcal{C} \cap \bar{\mathcal{N}} \cap \bar{\mathcal{M}}$ are not empty.

Results for Incomputable Numbers I

Fact

Martin-Löf random numbers are Borel absolutely-normal but neither computable nor Liouville numbers.

Example (Dilution (continued))

Let $\mathbf{x} = x_1 x_2 \cdots x_i \cdots \in A_b^\omega$ be Martin-Löf random and $\mathbf{y} := x_1 00 x_2 00 \cdots x_i 00 \cdots$. Then $0 < \underline{\kappa}(\mathbf{y}) = \frac{1}{3}$.

Thus $0.\mathbf{y}$ is neither Martin-Löf random nor computable nor a Liouville number. Obviously, $0.\mathbf{y}$ is also not normal.

Results for Incomputable Numbers II

Example (Non-random ω-words with $\underline{\kappa}(\mathbf{x}) = 1$ (continued))

Let $\mathbf{x} = x_1 x_2 \cdots x_i \cdots \in A_b^\omega$ be Martin-Löf random, and define

$$\mathbf{y}(i) := \begin{cases} 0 & \text{if } i \in \{2^n : n \in \mathbb{N}\}, \text{ and} \\ \mathbf{x}(i) & \text{otherwise.} \end{cases}$$

In view of $\underline{\kappa}(\mathbf{y}) = 1$ the real $\alpha = 0.\mathbf{y}$ is Borel absolutely-normal.

As in the preceding example, $\alpha = 0.\mathbf{y}$ is neither Martin-Löf random nor computable nor a Liouville number.

- The combinations $\bar{\mathcal{L}} \cap \bar{\mathcal{C}} \cap \mathcal{N} \cap \mathcal{M}$, $\bar{\mathcal{L}} \cap \bar{\mathcal{C}} \cap \bar{\mathcal{N}} \cap \bar{\mathcal{M}}$, and $\bar{\mathcal{L}} \cap \bar{\mathcal{C}} \cap \mathcal{N} \cap \bar{\mathcal{M}}$ are not empty.

Results for Computable Numbers

Example (Knight 1991)

Every rational number is computable but neither Liouville nor normal.

The computable irrational number $\alpha_b = \sum_{i \in \mathbb{N}} b^{-2^i}$ is non-Liouville but not normal in base b.

It is an *open problem* whether there exist computable, Borel absolutely-normal, non-Liouville numbers.

Theorem (Coons 2013)

The Stoneham number $F(1/2) = \sum_{i=1}^{\infty} 2^{-k^i} \cdot k^{-i}$ *(where* $k \in \mathbb{N}$ *is odd ,* $k \geq 3$*) is computable, normal in base* 2 *(but not in base* 6*), and, has irrationality exponent* $\mu(F(1/2)) = k < \infty$*.*

- The combination $\bar{\mathcal{L}} \cap \mathcal{C} \cap \bar{\mathcal{N}} \cap \bar{\mathcal{M}}$ is not empty, and $\bar{\mathcal{L}} \cap \mathcal{C} \cap \mathcal{N} \cap \bar{\mathcal{M}}$ might be not empty.

References

- Calude, C.S., Staiger, L.: *Liouville, Computable, Borel Normal and Martin-Löf Random Numbers*, CDMTCS-Report 448, 12/2013 file on: https://www.cs.auckland.ac.nz/research/groups/CDMTCS/researchreports

Books on Algorithmic Information Theory

- Calude, C.S.: *Information and Randomness. An Algorithmic Perspective*, 2nd ed., Springer, Berlin (2002).
- Downey, R., Hirschfeldt D.: *Algorithmic Randomness and Complexity*, Springer, Heidelberg (2010).
- Li, M., Vitányi, P.M.B.: *An Introduction to Kolmogorov Complexity and Its Applications*, Springer, Berlin (1993).
- Nies, A.: *Computability and Randomness*, Oxford Univ. Press, Oxford (2009).