

Constructive dimension and Hausdorff dimension: the case of exact dimension

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Outline

1 Preliminaries

Notation

Gambling strategies and super-martingales

HAUSDORFF dimension

Constructive dimension and KOLMOGOROV complexity

2 The original approach

HAUSDORFF '18

3 Exact Constructive Dimension

Lower bounds

Upper bounds

Sets of exact constructive dimension

4 Logarithmic Scale

Functions of the logarithmic scale

Notation: Strings and languages

Finite Alphabet $X = \{0, \dots, 2 - 1\}$, **cardinality** $|\{0, 1\}| = 2$

Finite strings (words) $w = x_1 \cdots x_n \in \{0, 1\}^*$, $x_i \in \{0, 1\}$

Length $|w| = n$

Languages $V, W \subseteq \{0, 1\}^*$

Infinite strings (ω -words) $\xi = x_1 \cdots x_n \cdots \in \{0, 1\}^\omega$

Prefixes of infinite strings $\xi[0..n] \in \{0, 1\}^*$, $|\xi[0..n]| = n$

ω -Languages $F \subseteq \{0, 1\}^\omega$

$\{0, 1\}^\omega$ as CANTOR space

Metric: $\rho(\eta, \xi) := \inf\{2^{-|w|} : w \in \mathbf{pref}(\eta) \cap \mathbf{pref}(\xi)\}$

Balls: $w \cdot \{0, 1\}^\omega = \{\eta : w \in \mathbf{pref}(\eta)\}$

Diameter: $\text{diam } w \cdot \{0, 1\}^\omega = 2^{-|w|}$

$\text{diam } F = \inf\{2^{-|w|} : F \subseteq w \cdot \{0, 1\}^\omega\}$

Open sets: $W \cdot \{0, 1\}^\omega = \bigcup_{w \in W} w \cdot \{0, 1\}^\omega$

Closure: $\mathcal{C}(F) = \{\xi : \mathbf{pref}(\xi) \subseteq \mathbf{pref}(F)\}$

Gambling strategies

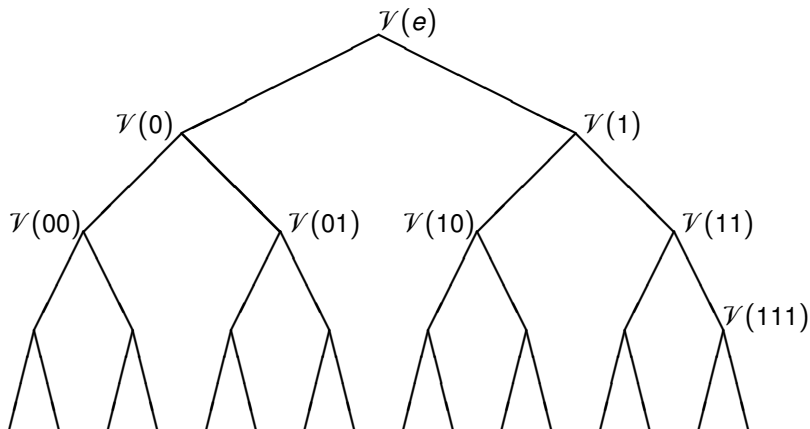
Our model:

- Playing head-and-tails against a binary sequence $\xi \in \{0, 1\}^\omega$
- Gambling strategy $\Gamma : \{0, 1\}^* \rightarrow [0, 1]$ (bet on outcome 1)
- yields a (super-)martingale $\mathcal{V}_\Gamma : \{0, 1\}^* \rightarrow \mathbb{R}_+$
- $\mathcal{V}_\Gamma(\xi[0..n])$ is the capital after the n th round

Fact (super-martingale property)

$$\mathcal{V}_\Gamma(w) \geq \sum_{x \in \{0,1\}} \frac{1}{2} \cdot \mathcal{V}_\Gamma(wx)$$

Gambling strategies: martingale \mathcal{V}



How much can you win: Order functions [SCHNORR'71]

Definition (Order function f and gauge function h)

$f : \mathbb{N} \rightarrow \mathbb{N}$ increasing

$h : (0, \infty) \rightarrow (0, \infty)$ right continuous and increasing

gauge function

order function

$$h : (0, \infty) \rightarrow (0, \infty)$$

$$f : \mathbb{N} \rightarrow \mathbb{N}$$

$$h(2^{-n}) = 2^{-n} \cdot f(n) \quad \longleftrightarrow$$

$$f(n)$$

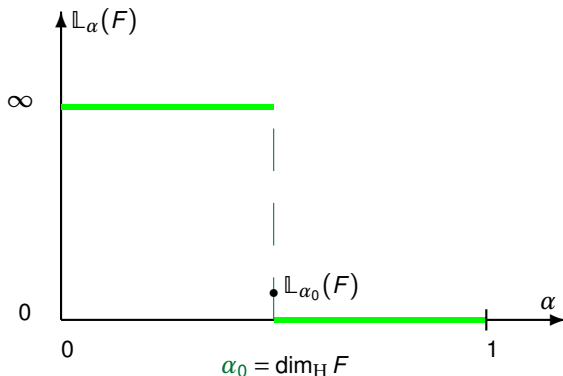
Definition (success set)

$$S_{c,h}[\mathcal{V}] := \left\{ \xi : \xi \in \{0, 1\}^\omega \wedge \limsup_{n \rightarrow \infty} \frac{\mathcal{V}(\xi[0..n])}{2^n \cdot h(2^{-n})} \geq c \right\}, \quad c \in (0, \infty) \cup \{\infty\}$$

“Classical” HAUSDORFF dimension

HAUSDORFF dimension of $F \subseteq \{0, 1\}^\omega$

$$\mathbb{L}_\alpha(F) := \lim_{n \rightarrow \infty} \inf \left\{ \sum_{v \in V} 2^{-\alpha \cdot |v|} : F \subseteq \bigcup_{v \in V} v \cdot \{0, 1\}^\omega \wedge \min_{v \in V} |v| \geq n \right\}$$



Relation to HAUSDORFF dimension

Let $S_{c,\alpha}[\mathcal{V}] := \left\{ \xi : \limsup_{n \rightarrow \infty} \frac{\mathcal{V}(\xi[0..n])}{2^n \cdot 2^{-\alpha \cdot n}} \geq c \right\}$ for $c \in (0, \infty) \cup \{\infty\}$

Lemma

For every super-martingale \mathcal{V} :

$$\dim_{\text{H}} S_{c,\alpha}[\mathcal{V}] \leq \alpha$$

Theorem ([LUTZ'03])

Let $F \subseteq \{0, 1\}^\omega$. Then

$$\dim_{\text{H}} F < \alpha \rightarrow \exists \mathcal{V} (F \subseteq S_{\infty,\alpha}[\mathcal{V}]) \rightarrow \dim_{\text{H}} F \leq \alpha.$$

Semi-computability

Definition (Left computable)

A (partial) mapping $\phi : \{0, 1\}^* \rightarrow \mathbb{R}$ is called **computably approximable from below** (*left computable*) : \iff

$$\{(w, q) : w \in \text{dom } \phi \wedge q \in \mathbb{Q} \wedge q < \phi(w)\}$$

is computably enumerable.

Similar: **computably approximable from above** (*right computable*)

Universal super-martingale

Definition (Continuous (or: Cylindrical) semi-measure)

$$\mu(w) \geq \mu(w0) + \mu(w1)$$

Theorem (LEVIN'70)

There is a universal left computable continuous semi-measure \mathbf{M} on $\{0, 1\}^$, that is, \mathbf{M} is left computable and for every left computable continuous semi-measure μ there is a constant c_μ such that*

$$\mu(w) \leq c_\mu \cdot \mathbf{M}(w) \quad \text{for all } w \in \{0, 1\}^* .$$

Theorem (LEVIN'70, SCHNORR'71)

There is a universal left computable super-martingale \mathcal{U} , e.g.

$$\mathcal{U}(w) := 2^{|w|} \cdot \mathbf{M}(w).$$

Constructive dimension [LUTZ'03]

Constructive dimension tries to measure for $\xi \in \{0, 1\}^\omega$ the exponent α for which

$$\mathcal{U}(\xi[0..n]) \approx 2^{\alpha \cdot n + o(n)}.$$

More precisely, $\mathcal{U}(\xi[0..n]) \geq_{i.o.} 2^{\alpha' \cdot n}$ for $\alpha' < \alpha$, and
 $\mathcal{U}(\xi[0..n]) \leq_{a.e.} 2^{\alpha'' \cdot n}$ for $\alpha'' > \alpha$,

that is [LEVIN'70, LUTZ'03],

Definition (Constructive dimension of ξ)

$$\underline{\kappa}(\xi) := 1 - \alpha = \liminf_{n \rightarrow \infty} \frac{-\log \mathbf{M}(\xi[0..n])}{n}$$

Corollary (LUTZ'03)

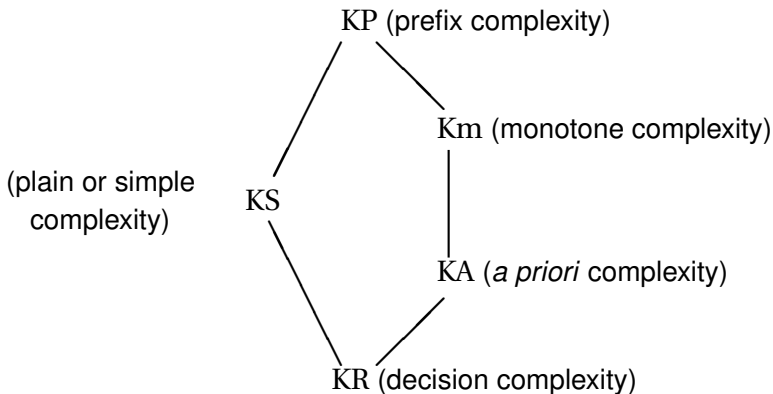
Let $F \subseteq \{0, 1\}^\omega$. Then

$$\sup\{\underline{\kappa}(\xi) : \xi \in F\} = \inf\{\alpha : F \subseteq S_{\infty, \alpha}[\mathcal{U}]\}.$$

KOLMOGOROV complexity – USPENSKY-SHEN-pentagon

a priori complexity

$$KA(w) := -\log \mathbf{M}(w)$$



The original approach [HAUSDORFF '18]

Classical

$$\mathbb{L}_\alpha(F) := \lim_{n \rightarrow \infty} \inf \left\{ \sum_{v \in V} (2^{-|v|})^\alpha : F \subseteq \bigcup_{v \in V} v \cdot \{0, 1\}^\omega \wedge \min_{v \in V} |v| \geq n \right\}$$

Original

$$\mathcal{H}^h(F) := \lim_{n \rightarrow \infty} \inf \left\{ \sum_{v \in V} h(2^{-|v|}) : F \subseteq \bigcup_{v \in V} v \cdot \{0, 1\}^\omega \wedge \min_{v \in V} |v| \geq n \right\}$$

where h is a *gauge function*, that is,

$h : (0, \infty) \rightarrow (0, \infty)$ is right continuous and non-decreasing,

The gauge functions for the “classical” HAUSDORFF dimension are

$$h_\alpha(t) = t^\alpha.$$

Exact HAUSDORFF dimension I

Lemma ([HAUSDORFF'18])

Let $h, h' : \mathbb{R}_+ \rightarrow \mathbb{R}_+$ be gauge functions and let $F \subseteq \{0, 1\}^\omega$.

- 1 If $h(t) \leq c \cdot h'(t)$ then $\mathcal{H}^h(F) \leq c \cdot \mathcal{H}^{h'}(F)$.
- 2 If $\lim_{t \rightarrow 0} \frac{h(t)}{h'(t)} = 0$ then $\mathcal{H}^{h'}(F) < \infty$ implies $\mathcal{H}^h(F) = 0$,
and $\mathcal{H}^h(F) > 0$ implies $\mathcal{H}^{h'}(F) = \infty$.

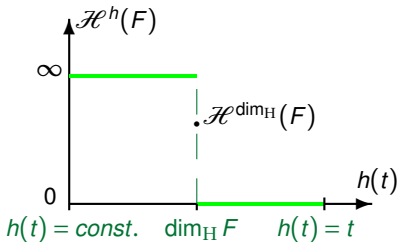
Quasi-ordering of gauge functions (Speed of converging to 0)

Largest $h(t) = t$

Ordering $h(t') < h(t)$ if and only if $\lim_{t \rightarrow 0} \frac{h(t)}{h'(t)} = 0$,
e.g. the exponential functions $h(t) = t^\alpha, 0 \leq \alpha \leq 1$

Smallest $h(t) = \text{const.} > 0$

Exact HAUSDORFF dimension II



Definition (Exact HAUSDORFF dimension)

We refer to a gauge function h as an **exact Hausdorff dimension function** for $F \subseteq \{0, 1\}^\omega$ provided

$$\mathcal{H}^{h'}(F) = \begin{cases} \infty, & \text{if } \lim_{t \rightarrow 0} \frac{h(t)}{h'(t)} = 0, \text{ and} \\ 0, & \text{if } \lim_{t \rightarrow 0} \frac{h'(t)}{h(t)} = 0. \end{cases}$$

Martingale characterisation of exact HAUSDORFF dimension

Definition (Success set)

$$S_{c,h}[\mathcal{V}] := \left\{ \xi : \limsup_{n \rightarrow \infty} \frac{\mathcal{V}(\xi[0..n])}{2^n \cdot h(2^{-n})} \geq c \right\} \text{ for } c \in (0, \infty) \cup \{\infty\}$$

Theorem

h is an exact Hausdorff dimension function for $F \subseteq \{0, 1\}^\omega$: \iff

- 1 for all gauge functions h' with $\lim_{t \rightarrow 0} \frac{h'(t)}{h(t)} = 0$ there is a (super-)martingale \mathcal{V} such that $F \subseteq S_{\infty, h'}[\mathcal{V}]$, and
- 2 $F \not\subseteq S_{\infty, h'}[\mathcal{V}]$ for all (super-)martingales \mathcal{V} and all gauge functions h'' with $\lim_{t \rightarrow 0} \frac{h(t)}{h''(t)} = 0$.

Exact constructive dimension

Analogously to the **martingale characterisation of the Hausdorff dimension** we set:

Definition

We refer to a gauge function h as an **exact constructive dimension function** for $F \subseteq \{0, 1\}^\omega$ provided

- 1 $F \subseteq S_{\infty, h'}[\mathcal{U}]$ for all gauge functions h' with $\lim_{t \rightarrow 0} \frac{h'(t)}{h(t)} = 0$, and
- 2 $F \not\subseteq S_{\infty, h''}[\mathcal{U}]$ for all gauge functions h'' with $\lim_{t \rightarrow 0} \frac{h(t)}{h''(t)} = 0$.

Theorem (Exact dimension for $\{\xi\}$)

The function h_ξ defined by $h_\xi(2^{-n}) := 2^{-n} \cdot \mathcal{U}(\xi[0..n]) = \mathbf{M}(\xi[0..n])$ is an exact constructive dimension function for the set $\{\xi\}$.

RYABKO's result: Large sets contain complex ω -words

Theorem (RYABKO'84, "classical case")

For $\alpha \in [0, 1]$ it holds $\alpha = \dim_{\text{H}}\{\xi : \xi \in \{0, 1\}^{\omega} \wedge \underline{\kappa}(\xi) \leq \alpha\}$.

Theorem (St'93, "classical case")

If $F \subseteq \{0, 1\}^{\omega}$ and $\mathbb{L}_{\alpha}(F) > 0$ then there is a $\xi \in F$ such that

$$\liminf_{n \rightarrow \infty} \frac{\text{KS}(\xi[0..n])}{n} \geq_{\text{a.e.}} \alpha \cdot n - (1 + \varepsilon) \log n.$$

Theorem (Lower KA-bound, Mielke'09)

Let $F \subseteq \{0, 1\}^{\omega}$, h be a gauge function and $\mathcal{H}^h(F) > 0$.

Then for every $c > 0$ with $\mathcal{H}^h(F) > c \cdot \mathbf{M}(e)$ there is a $\xi \in F$ such that

$$\text{KA}(\xi[0..n]) \geq_{\text{a.e.}} -\log h(2^{-n}) + c.$$

Complexity bounds

Fact

$$\{\xi : \exists c(\text{KA}(\xi[0..n]) \leq_{\text{i.o.}} -\log h(2^{-n}) + c)\} = \bigcup_{c' \in (0, \infty)} S_{c', h}[\mathcal{U}]$$

Corollary (to Lower KA-bound)

Let h, h' be gauge functions such that $\lim_{t \rightarrow 0} \frac{h'(t)}{h(t)} = 0$. Then

- ① $\{\xi : \exists c(\text{KA}(\xi[0..n]) \leq_{\text{i.o.}} -\log h(2^{-n}) + c)\} \subseteq S_{\infty, h'}[\mathcal{U}]$, and
- ② $\mathcal{H}^{h'}(\{\xi : \exists c(\text{KA}(\xi[0..n]) \leq_{\text{i.o.}} -\log h(2^{-n}) + c)\}) = 0$.

Exact dimension function: an example

Example

$$F := \{x_1 x_2 \cdots x_i \cdots \mid x_i \in \{0, 1\} \wedge \forall j (x_{2^j} = 0)\}$$

F has classical Hausdorff dimension $\dim_{\text{H}} F = 1$ but does not contain any random sequence.

The exact Hausdorff dimension is

$$\dim_{\text{H}} F = \lceil h(t) = t \cdot \log \frac{1}{t} \rceil.$$

Observe $-\log h(2^{-n}) = n - \log n$.

Upper bounds: “Classical case” [St’98]

Definition (Σ_2 -sets)

A subset $F \subseteq \{0, 1\}^\omega$ is Σ_2 -definable if there is a computable set $W \subseteq \mathbb{N} \times \{0, 1\}^*$ such that

$$\xi \in F \longleftrightarrow \exists m \forall n ((m, \xi[0..n]) \in W).$$

Theorem

If $F \subseteq \{0, 1\}^\omega$ is Σ_2 -definable and $\alpha \geq \dim_{\text{H}} F$ is a right computable real then there is a computable $V \subseteq \{0, 1\}^$ such that $F \subseteq V^\delta$ and $\sum_{V \in V} 2^{-|V|} < \infty$.*

Corollary

If $F_i \subseteq \{0, 1\}^\omega$, $i \in \mathbb{N}$, are Σ_2 -definable then $\underline{\kappa}(\xi) \leq \dim \bigcup_{i \in \mathbb{N}} F_i$ for $\xi \in \bigcup_{i \in \mathbb{N}} F_i$.

Upper bound on prefix complexity KP

Lemma (REIMANN'04)

Let $F \subseteq \{0, 1\}^\omega$ and h be a gauge function. Then $\mathcal{H}^h(F) = 0$ if and only if there is a $V \subseteq \{0, 1\}^*$ such that $F \subseteq V^\delta := \{\xi : |\text{pref}(\xi) \cap V| = \infty\}$ and $\sum_{v \in V} h(2^{-|v|}) < \infty$.

Theorem

If $F \subseteq \{0, 1\}^\omega$ is Σ_2 -definable and h is a right computable gauge function such that $\mathcal{H}^h(F) = 0$ then there are a non-decreasing function $\bar{h} : \{2^{-i} : i \in \mathbb{N}\} \rightarrow \mathbb{Q}$ and a computable $V \subseteq \{0, 1\}^*$ such that

- 1 $\bar{h}(2^{-i}) \geq h(2^{-i})$ for $i \in \mathbb{N}$,
- 2 $\sum_{v \in V} \bar{h}(2^{-|v|}) < \infty$ and $F \subseteq V^\delta$, and
- 3 $\text{KP}(\xi[0..n]) \leq_{\text{i.o.}} -\log_r h(r^{-n}) + O(1)$ for all $\xi \in F$.

Upper bounds: Computable dimension

Theorem (“Classical case”, St’98)

If $F \subseteq \{0, 1\}^\omega$ is Σ_2 -definable and $\alpha \geq \dim_{\text{H}} F$ is a right computable real then there is a computable martingale \mathcal{V} such that $F \subseteq S_{\infty, \alpha}[\mathcal{V}]$.

Theorem

For every Σ_2 -definable $F \subseteq \{0, 1\}^\omega$ and every computable gauge function $h: \mathbb{Q} \rightarrow \mathbb{R}$ such that $\mathcal{H}^h(F) = 0$ there is a computable martingale \mathcal{V} such that $F \subseteq \bigcup_{c>0} S_{c, h}[\mathcal{V}]$.

Dilution functions

Modulus function: $g : \mathbb{N} \rightarrow \mathbb{N}$ strictly monotone, that is, $g(n+1) > g(n)$

Example [Dilution function with $|\varphi(w)| = g(|w|)$]

$\varphi : \{0, 1\}^* \rightarrow \{0, 1\}^*$

$\varphi(e) := 0^{g(0)}$ and

$\varphi(wx) := \varphi(w) \cdot x \cdot 0^{g(n+1)-g(n)-1}$

Definition (Dilution function)

For every $v \in \mathbf{pref}(\varphi(\{0, 1\}^*))$ there are $w_v \in \{0, 1\}^*$ and $x_v \in \{0, 1\}$ such that

$$\varphi(w_v) \sqsubset v \sqsubseteq \varphi(w_v \cdot x_v) \quad \wedge \quad \forall y (y \in \{0, 1\} \wedge y \neq x_v \rightarrow v \not\sqsubseteq \varphi(w_v \cdot y))$$

HAUSDORFF measure of diluted sets

Theorem

Let $g : \mathbb{N} \rightarrow \mathbb{N}$ be a strictly increasing function, φ a corresponding dilation function and $h : (0, \infty) \rightarrow (0, \infty)$ be a gauge function. Then

- 1 $\mathcal{H}^h(\overline{\varphi}(\{0, 1\}^\omega)) \leq \liminf_{n \rightarrow \infty} \frac{h(2^{-g(n)})}{2^{-n}}$, and
- 2 if $c \cdot 2^{-n} \leq_{\text{a.e.}} h(2^{-g(n)})$ then $c \leq \mathcal{H}^h(\overline{\varphi}(\{0, 1\}^\omega))$.

Corollary

If $c \cdot 2^{-n} \leq_{\text{a.e.}} h(2^{-g(n)}) \leq c' \cdot 2^{-n}$ then $c \leq \mathcal{H}^h(\overline{\varphi}(\{0, 1\}^\omega)) \leq c'$.

Dilution: an existence condition for modulus functions

Lemma (Sufficient condition)

If a gauge function h is upwardly convex (or \cap -convex) on some interval $(0, \varepsilon)$ and $\lim_{t \rightarrow 0} h(t) = 0$ then there is an $n_0 \in \mathbb{N}$ such that for all $n \geq n_0$ there is an $m \in \mathbb{N}$ satisfying

$$2^{-n} < h(2^{-m}) \leq 2^{-n+1}.$$

In particular, there are a modulus function $g : \mathbb{N} \rightarrow \mathbb{N}$ and constants c_0, c_1 such that

$$0 < c_0 \leq \liminf_{n \rightarrow \infty} \frac{h(2^{-g(n)})}{2^{-n}} \leq \limsup_{n \rightarrow \infty} \frac{h(2^{-g(n)})}{2^{-n}} \leq c_1$$

If, moreover, $h : \mathbb{Q} \rightarrow \mathbb{R}$ is a computable gauge function then also $g : \mathbb{N} \rightarrow \mathbb{N}$ can be chosen to be computable.

Exact complexity bound

Theorem (St'09)

Let $\varphi : \{0, 1\}^* \rightarrow \{0, 1\}^*$ be a computable dilution function with modulus function $g : \mathbb{N} \rightarrow \mathbb{N}$. Then

$|\text{KA}(\overline{\varphi}(\xi)[0..g(n)]) - \text{KA}(\xi[0..n])| \leq O(1)$ for all $\xi \in \{0, 1\}^\omega$ and all $n \in \mathbb{N}$.

Theorem

Let $h : \mathbb{Q} \rightarrow \mathbb{R}$ be a computable gauge function such that for all $n \geq n_0$ there is an $m \in \mathbb{N}$ with $2^{-n} < h(2^{-m}) \leq 2^{-n+1}$. Then

- 1 $\mathcal{H}^h(\{\xi : \exists c(\text{KA}(\xi[0..n]) \leq_{\text{a.e.}} -\log h(2^{-n}) + c)\}) > 0$, and
- 2 h is an exact dimension function for the sets $\{\xi : \exists c(\text{KA}(\xi[0..n]) \leq_{\text{i.o.}} -\log h(2^{-n}) + c)\}$ and $\{\zeta : \exists c(\text{KA}(\zeta[0..n]) \leq_{\text{a.e.}} -\log h(2^{-n}) + c)\}$.

Functions of the Logarithmic Scale

Definition (Functions of the logarithmic scale)

$$h_{(p_0, \dots, p_k)}(t) = t^{p_0} \cdot \prod_{i=1}^k \left(\log^i \frac{1}{t} \right)^{-p_i}$$

where $\log^i t := \max\{1, \underbrace{\log_2 \dots \log_2 t}_{i \text{ times}}\}$.

Definition (Generalised HAUSDORFF Dimension)

$$\begin{aligned} \dim_{\mathbb{H}}^{(k)} F &:= \sup_{<_{\text{lex}}} \{(p_0, \dots, p_k) : \mathcal{H}^{h_{(p_0, \dots, p_k)}}(F) = \infty\} \\ &= \inf_{<_{\text{lex}}} \{(p_0, \dots, p_k) : \mathcal{H}^{h_{(p_0, \dots, p_k)}}(F) = 0\} \end{aligned}$$

Upper bound

Let $h_{(p_0, \dots, p_k)}$, $k > 0$, be a function of the logarithmic scale. We define $\beta_h := \log h_{(p_0, \dots, p_{k-1})}$. Observe that

$$\begin{aligned}\beta_h(2^{-n}) &= p_0 \cdot n - \sum_{i=1}^{k-1} p_i \cdot \log^i n \quad \text{and} \\ \log h_{(p_0, \dots, p_k)}(2^{-n}) &= p_0 \cdot n - \sum_{i=1}^{k-1} p_i \cdot \log^i n - p_k \cdot \log^k n\end{aligned}$$

Theorem (MIELKE'10)

Let $k \geq 0$, (p_0, \dots, p_k) be a $(k+1)$ -tuple and $h_{(p_0, \dots, p_k)}$ be a function of the logarithmic scale. Then

$$\dim_{\mathbb{H}}^{(k)} \left\{ \xi : \xi \in \{0, 1\}^{\omega} \wedge \liminf_{n \rightarrow \infty} \frac{\text{KA}(\xi[0..n]) - \beta_h(2^{-n})}{\log^k n} < p_k \right\} \leq (p_0, \dots, p_k).$$

Lower bound

Theorem (MIELKE'10,St)

Let $k > 0$, (p_0, \dots, p_k) be a $(k + 1)$ -tuple where p_0, \dots, p_{k-1} are computable reals. Then

$$\dim_{\mathbb{H}}^{(k)} \left\{ \xi : \xi \in \{0, 1\}^{\omega} \wedge \liminf_{n \rightarrow \infty} \frac{\text{KA}(\xi[0..n]) - \beta_h(2^{-n})}{\log^k n} < p_k \right\} = (p_0, \dots, p_k)$$

for $h = h_{(p_0, \dots, p_k)}$.

Why our bounds don't match

“Inexact” case: approximation of real by computable reals

For every real number α and every $\varepsilon > 0$ there are computable reals α_0, α_1 such that $|\alpha_1 - \alpha_0| < \varepsilon$ and $\alpha_0 \leq \alpha \leq \alpha_1$.

Example: logarithmic scale

If there is a computable function $h: \mathbb{Q} \rightarrow \mathbb{R}$ such that

$$t^{p_0} \leq h(t) \leq t^{p_0} \cdot \log \frac{1}{t} \text{ for } t \in (0, 1) \cap \mathbb{Q}$$

then p_0 is computable.

Counter-example

RYABKO's theorem is independent of the complexity. That is, we can replace $\mathbb{K}A$ by other complexities, e.g. by plain KOLMOGOROV complexity $\mathbb{K}S$.

What about our theorem?

Example [Oscillation of the plain complexity $\mathbb{K}S$]

It is known that $\mathbb{K}S(\xi[0..n]) \leq_{i.o.} n - \log n + O(1)$. Thus

$$\left\{ \xi : \xi \in \{0, 1\}^\omega \wedge \liminf_{n \rightarrow \infty} \frac{\mathbb{K}S(\xi[0..n]) - n}{\log n} < \varepsilon - 1 \right\} = \{0, 1\}^\omega \text{ for all } \varepsilon > 0$$

but

$$\dim_{\mathbb{H}}^{(1)} \left\{ \xi : \xi \in \{0, 1\}^\omega \wedge \liminf_{n \rightarrow \infty} \frac{\mathbb{K}S(\xi[0..n]) - n}{\log n} < \varepsilon - 1 \right\} = 1 >_{\text{lex}} (1, \varepsilon - 1).$$